The Convergence Properties of the BLP (1995) Contraction Mapping and Alternative Algorithms in R

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A number of real-world estimation problems require the solution of large sets of nonlinear equations. For some of these problems, fixed-point iteration schemes have been proposed, such as the Berry *et al.* (1995, BLP) Contraction Mapping algorithm used in estimating random coefficients logit models of demand for differentiated products. This is a fixed-point problem in J products by T markets, where one must invert a demand system to uncover a vector $\boldsymbol{\delta} \in \mathbb{R}^{J\times T}$ that captures the "mean" utility for each product $j = 1, \ldots, J$ in each market $t = 1, \ldots, T$, and that equates predicted market shares $\hat{s}$ with the actual observed market shares S, as in $\hat{s}(\boldsymbol{\delta}, \boldsymbol{\sigma}) = S$. With $\boldsymbol{\sigma}$ the vector of standard deviations of the distributions of individual tastes for product characteristics, the BLP Contraction Mapping thus involves computing $\boldsymbol{\delta}(\boldsymbol{\sigma}) = \hat{s}^{-1}(S; \boldsymbol{\sigma})$ using the following iterative scheme:

1. For each value of $\boldsymbol{\sigma}$, compute the next value for $\boldsymbol{\delta}$ using

$$\boldsymbol{\delta}^{h+1} = \boldsymbol{\delta}^{h} + \log(S) - \log\left(\hat{s}(\boldsymbol{\delta}^{h}, \boldsymbol{\sigma})\right). \tag{1}$$

2. Stop if $\|\boldsymbol{\delta}^{h+1} - \boldsymbol{\delta}^{h}\| \leq \epsilon$, where $\|\cdot\|$ can be either L_2 or L_∞ and $\epsilon =$ `1e-09` is the tolerance level.

Given its linear rate of convergence and the size of $\boldsymbol{\delta}$ (typically exceeding 1000 observations), this is a time-consuming procedure. By reformulating the fixed-point problem as a nonlinear rootfinding problem, this paper introduces alternative methods to speed up the convergence process, specifically (1) the classical Newton-Raphson (N-R) method, (2) the Broyden secant method (see e.g. Dennis and Schnabel, 1983), and (3) the derivative-free spectral agorithm (DF-SANE). In a Monte Carlo study representing various scenarios, we use the **BB** (Varadhan and Gilbert, 2009) and **nleqslv** (Hasselman, 2009) packages for the implementation in R (R Development Core Team, 2009). We find that DF-SANE is more than three times faster than the BLP Contraction Mapping and achieves convergence in nearly 100% of the simulations runs throughout different scenarios. As suspected for this large system of nonlinear equations, numerical N-R is very slow to converge compared with BLP. The analytical N-R method is substantially faster, with speed of convergence similar to DF-SANE.

A central issue in accelerating iterative algorithms is the trade-off between stability (global convergence) and speed. If the cost of increasing speed is measured in decreased stability, compared to other acceleration schemes the SQUAREM algorithm (Varadhan and Roland, 2008) pays a much smaller price while procuring a decent gain in speed. We are therefore also exploring this globally convergent algorithm, which provides for explicit trade-off between speed and stability, for accelerating the BLP Contraction Mapping.

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