Threshold cointegration in R with package tsDyn

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8 July 2009

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Outline

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Background

- Non-stationnary variables with unit root: I(1)
- Spurious regression when I(1) regressed on I(1):
  - $R^2 \longrightarrow 1$
  - Statistical dependance among independant variables
  - **Wrong conclusions!**

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Cointegration

Definition (Cointegration (Engle, Granger 1982))

If two (or more) variables are *non-stationary*,
but there exist a linear combination of them which is *stationary*,
there are said to be *cointegrated*

Example

X and Y as I(1),
Take $X_t - aY_t = \varepsilon_t$
X and Y cointegrated $\Leftrightarrow \varepsilon$ is I(0)

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Interest of linear cointegration

- Stable **long-run relationship** between random walk variables.
- **Error-correction mechanisms** pushing deviations back towards the long-run equilibirum.

Example (VECM model with cointegrated variables)

$$\begin{pmatrix} \Delta X_t \\ \Delta Y_t \end{pmatrix} = \begin{pmatrix} 0.02 \\ -0.01 \end{pmatrix} + \begin{pmatrix} -0.01 \\ 0.08 \end{pmatrix} ECT_{t-1} + \begin{pmatrix} 0.04 & 0.02 \\ 0.31 & 0.07 \end{pmatrix} \begin{pmatrix} \Delta X_{t-1} \\ \Delta Y_{t-1} \end{pmatrix}$$

Where ECT (error-correction term) represents deviations from the long-run relationship $ECT_{t-1} = Y_t - bX_t$

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Whith $ECT_{t-1} = Y_t - bX_t$

The assumption of linearity

Implicit assumption: every small/big deviation from equilibirum leads to **instantaneous correction**.

But economic theory suggests:

- Transaction costs (no adjustment when: deviations $<$ transaction costs)
- Stickiness of the price
- Asymetries: $+/-$ deviations don't lead to same effect

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The threshold autoregressive (TAR) model

Linear model:

$$\text{AR}: \varepsilon_t = \rho\varepsilon_{t-1} + u_t$$

Regime-specific dynamics in the Threshold Autoregressive (TAR) model:

$$\text{TAR(2)}: \qquad \varepsilon_t = \begin{cases} \rho^L \varepsilon_{t-1} + u_t & \text{if} \quad \varepsilon_{t-1} \leq 0 \\ \rho^H \varepsilon_{t-1} + u_t & \text{if} \quad 0 \leq \varepsilon_{t-1} \end{cases}$$

$$\text{TAR(3)}: \qquad \varepsilon_t = \begin{cases} \rho^L \varepsilon_{t-1} + u_t & \text{if} \quad \varepsilon_{t-1} \leq \theta^L \\ \rho^M \varepsilon_{t-1} + u_t & \text{if} \quad \theta^L \leq \varepsilon_{t-1} \leq \theta^H \\ \rho^H \varepsilon_{t-1} + u_t & \text{if} \quad \theta^H \leq \varepsilon_{t-1} \end{cases}$$

Stationarity condition:

- $|\rho^L| < 1, |\rho^H| < 1$
- $|\rho^M| < \infty$ (non-stationarity of middle regime doesn't affect stationarity of whole proces)

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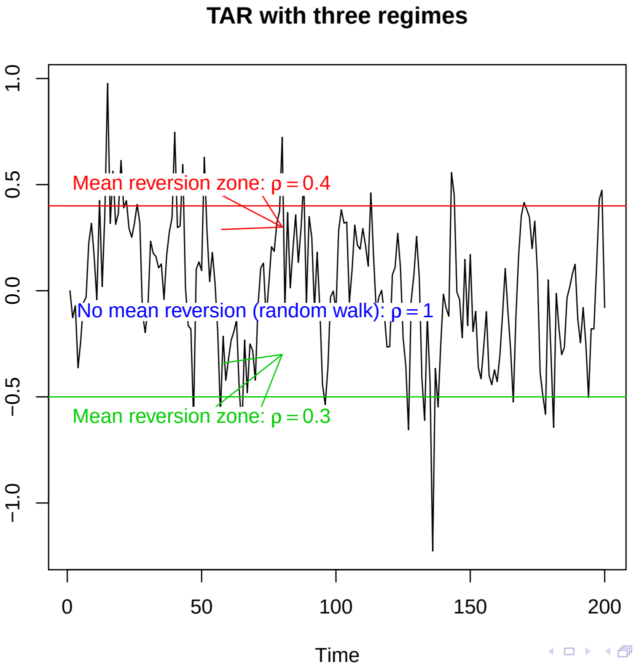
TAR with three regimes
Mean reversion zone: ρ = 0.4
No mean reversion (random walk): ρ = 1
Mean reversion zone: ρ = 0.3
1.0
0.5
0.0
−0.5
−1.0
0
50
100
150
200
Time

Threshold cointegration

Definition (Threshold cointegration)

If two (or more) variables are I(1),
but there exist a linear combination of them which is "*threshold stationary*",
there are said to be "*threshold cointegrated*"

Two main features:

- Allows no-adjustment band
- Allows asymetries: different +/- adjustment speeds ($\rho^H \neq \rho^L$)

Threshold effects in:

- Long-run (LR) relationship
- VECM

Threshold effects in the VECM

- Linear case

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- Note:
  - lags can also be regime specific
  - Same feature: adjustment band, asymetries

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Areas of application

- Macroeconomics questions
  - Law of one price (LOP)
  - Purchasing power parity
  - Exchange rate pass-through
  - Fisher effect: nominal interest rates and inflation
  - Usual macro: price, interest rate, income
- Price transmission studies
  - Vertically: market chains, numerous studies for agricultural products, oil
  - Horizontally: market integration, similar to LOP
- Financial markets
  - Term interest theory
  - Stock Prices and Dividends
  - Futures market
  - Various arbitrage markets

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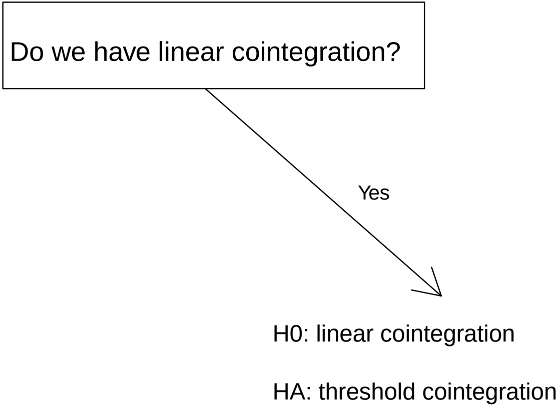
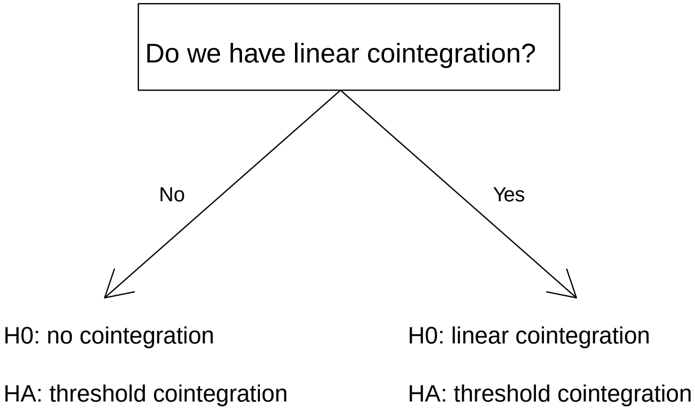
Implementation in R: package tsDyn

- Testing
- Estimation

Testing


Do we have linear cointegration?
Yes
H0: linear cointegration
HA: threshold cointegration

Testing


Do we have linear cointegration?
No
Yes
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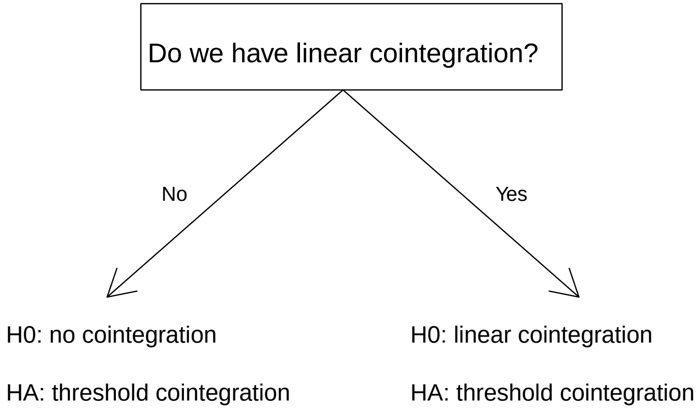
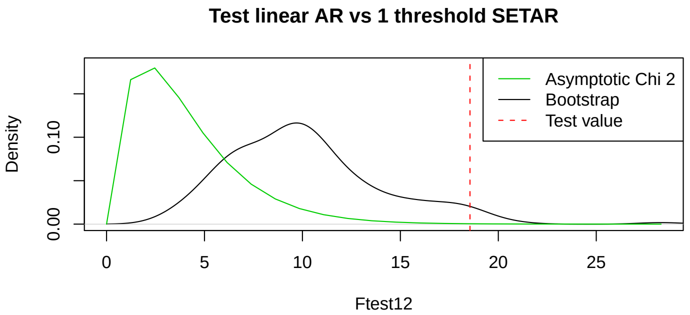
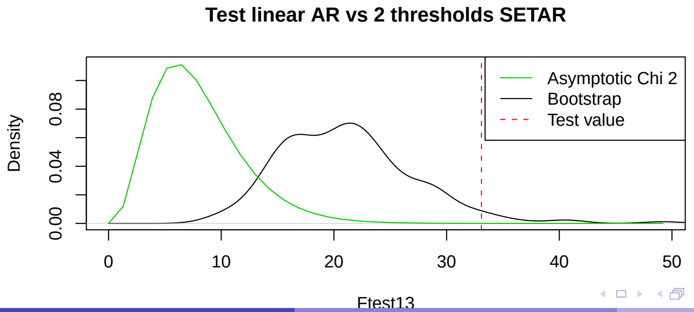


Interesting case

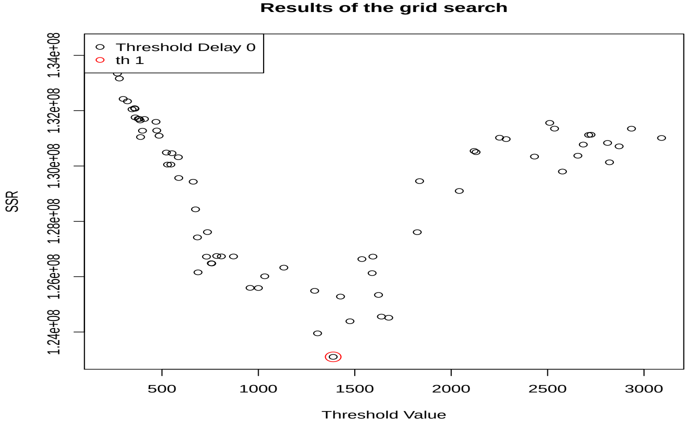
Case of no linear but threshold cointegration!


Test linear AR vs 1 threshold SETAR
Density
0.00
0.10
0
5
10
15
20
25
Ftest12
Asymptotic Chi 2
Bootstrap
Test value


Test linear AR vs 2 thresholds SETAR
Density
0.00
0.04
0.08
0
10
20
30
40
50
Ftest13
Asymptotic Chi 2
Bootstrap
Test value

Estimation of the threshold

Estimation: grid search in the range of all possible values



Summary

Threshold cointegration answers the following questions:

- Is there a long-run relationship? (Generalization of linear cointegration)
- Is there a *no arbitrage* band?
- Are there asymmetries, different adjustment speeds when increase or decrease?

Further readings

- Package vignette
- Working-paper: *Threshold cointegration: overview and implementation in R*

Thank you.

If because of the stress I spoke to fast

...and have some time left:

Additional features:

- Simulation of TAR, (T)VAR and (T)VECM
- Other representations of output compared to vars
- toLatex() function for VAR and VECm