A Tale of Two Theories:

Reconciling
random matrix theory and shrinkage estimation
as methods for covariance matrix estimation

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Disclaimer

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Overview

- Motivation
- Random Matrix Theory
- Shrinkage Estimation
- Measuring Effectiveness
  - Kullback-Leibler distance
  - Financial measures
- Reconciliation

Motivation

- Sample covariance != true covariance matrix
- Estimation error is large when !(T >> N)
  - Large portfolios
  - Monthly time frame
- Need a good estimate of covariance matrix

Approaches

Physics: Random matrix theory

- Eigenvalue distribution

Approaches

Physics: Random matrix theory

- Eigenvalue distribution
- Null hypothesis

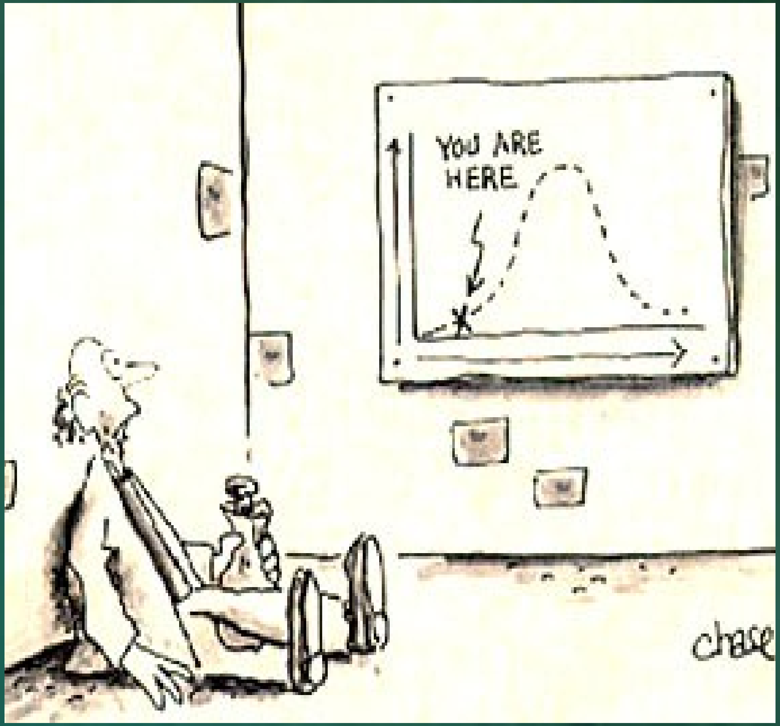
Approaches

Physics: Random matrix theory

- Eigenvalue distribution
- Null hypothesis
- Remove noise component

Approaches

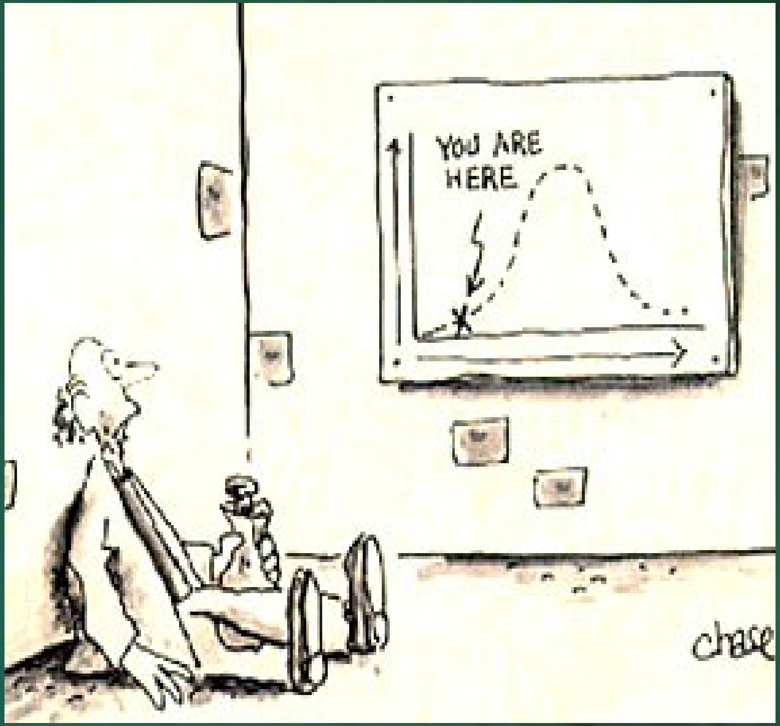
Statistics: Shrinkage Estimation



- Central limit theorem

Approaches

Statistics: Shrinkage Estimation

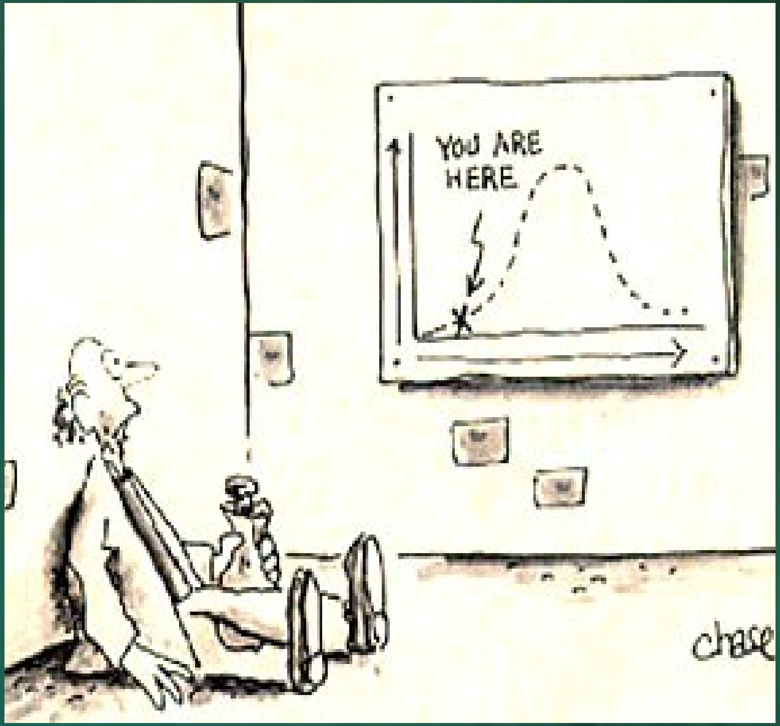


- Central limit theorem
- Weighted average

$\alpha F + (1-\alpha) S$

Approaches

Statistics: Shrinkage Estimation

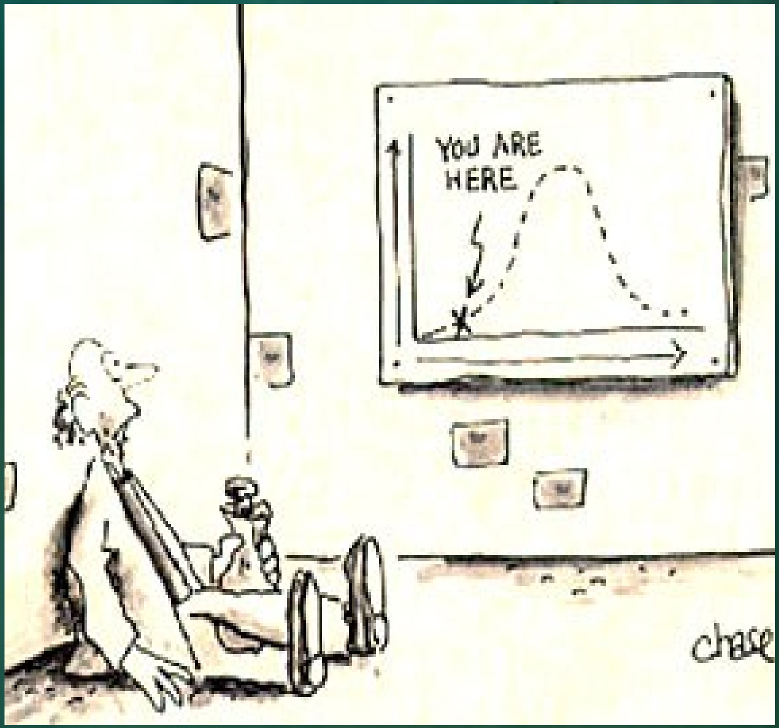


- Central limit theorem
- Weighted average

  $\alpha F + (1-\alpha) S$
- Reduced estimation error

Approaches

Which is Right?



Random Matrix Theory

- Eigenvalue distribution of random matrices is defined by the Marcenko-Pastur limit

$$\rho(\lambda) = \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda_{min} - \lambda)}}{\lambda}$$

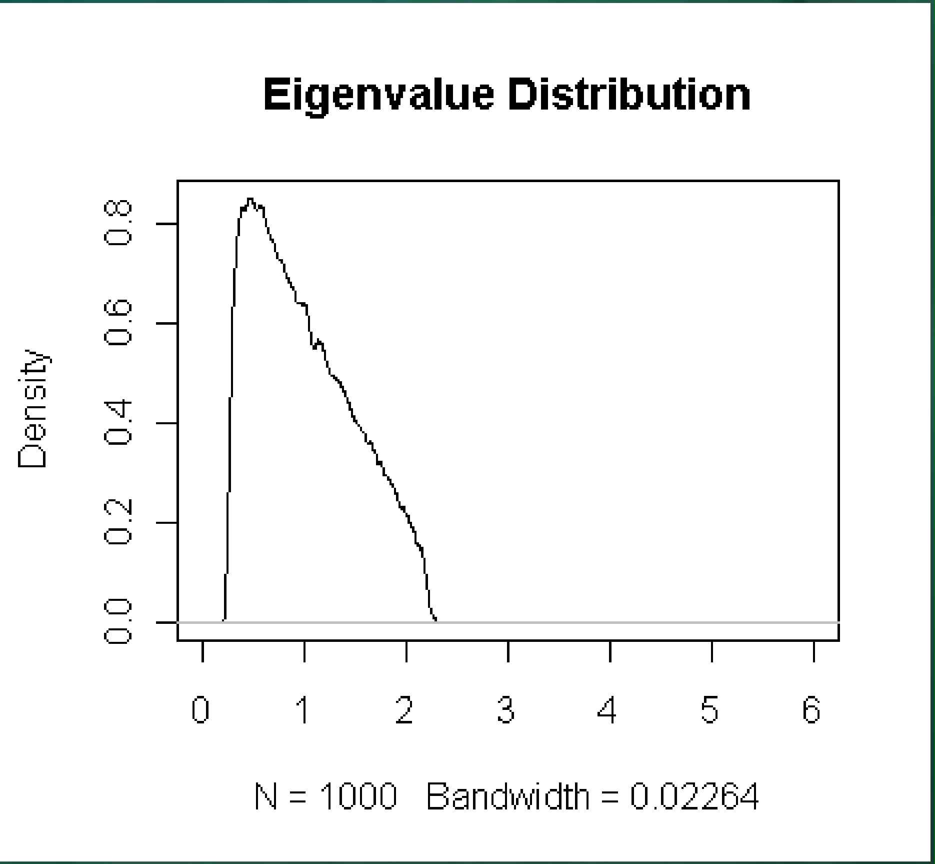
$$\lambda_{max/min} = \sigma^2 \left(1 \pm \sqrt{\tfrac{1}{Q}}\right)^2$$

- Sample correlation matrices can be filtered to remove this noise
- The reconstructed matrix is then used in portfolio optimization

Random Matrix Theory

Marcenko-Pastur Distributions

- Random matrix with normal distribution; N=1000, T=4000
- Random matrix with normal distribution; N=250, T=1000
- Random matrix with normal distribution; N=50, T=200



Random Matrix Theory

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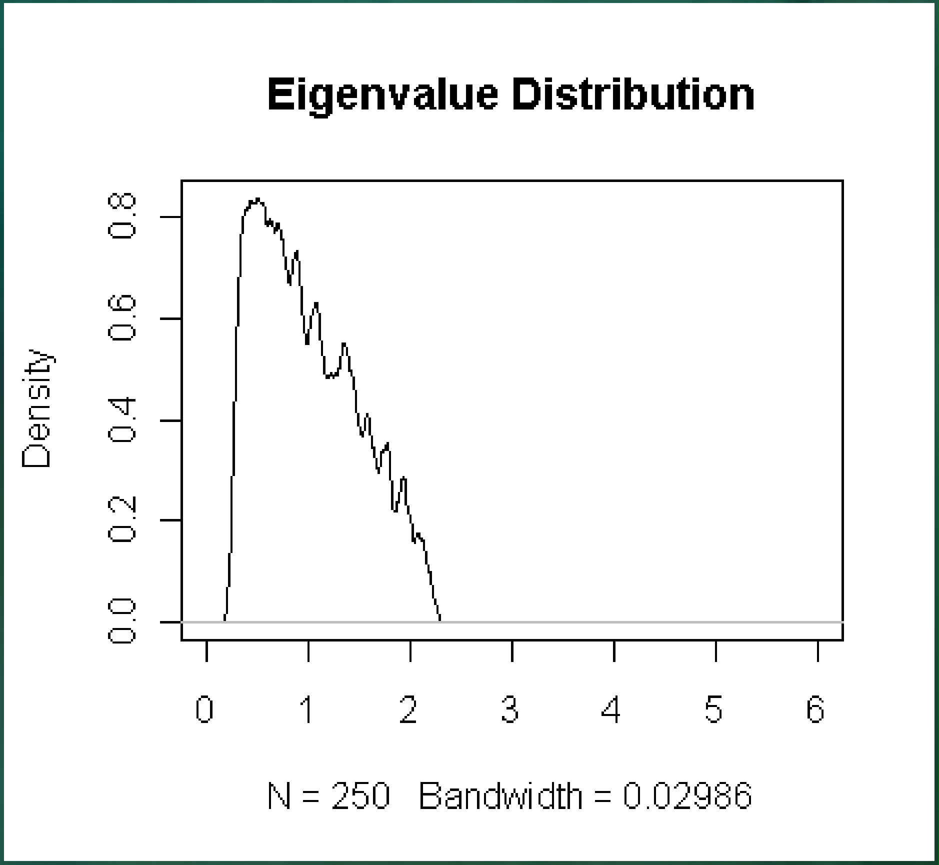

Random Matrix Theory

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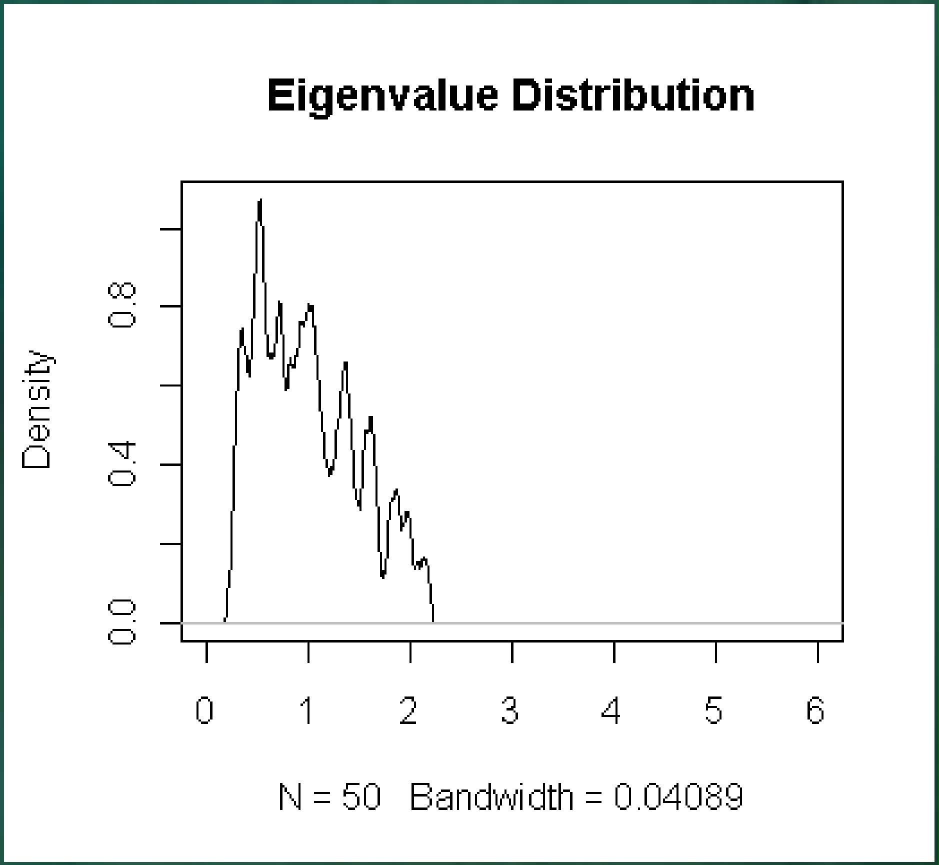
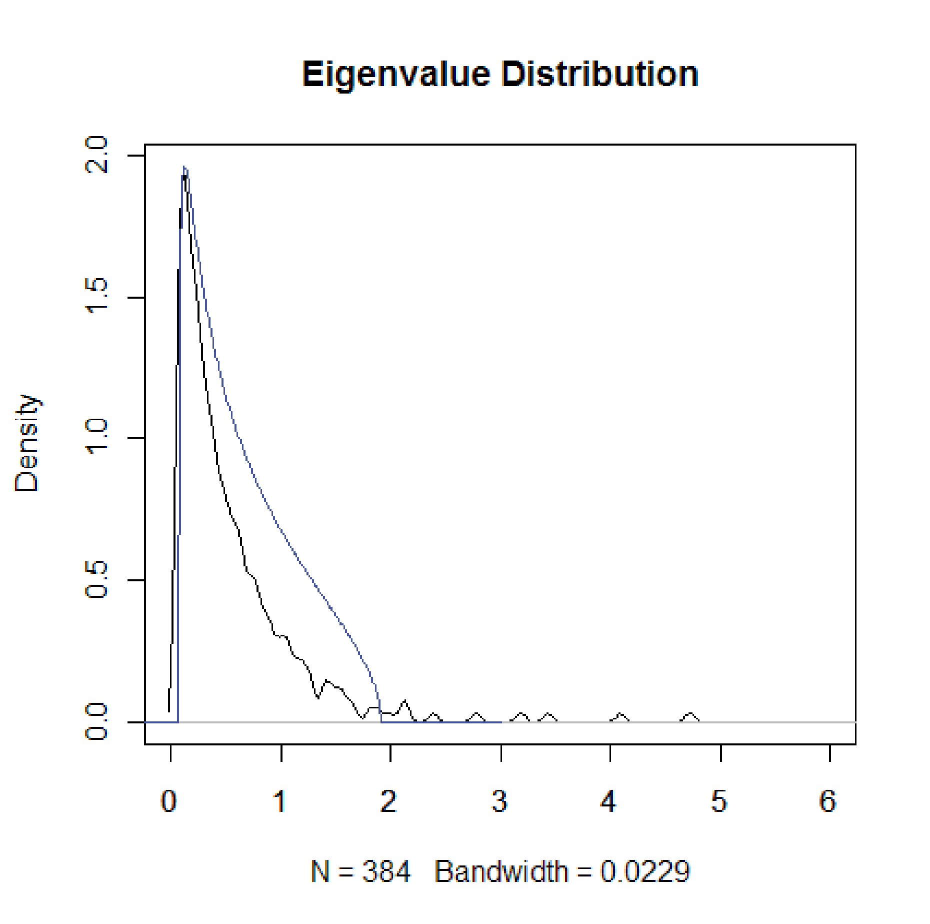


Random Matrix Theory

Fitting the Null Hypothesis

- Daily S&P 500; N=384, T=1200
- Daily S&P 500 subset; N=75, T=200
- Shuffled S&P 500; N=75, T=200

$Q = 2.072958$
$\sigma = 0.8152044$

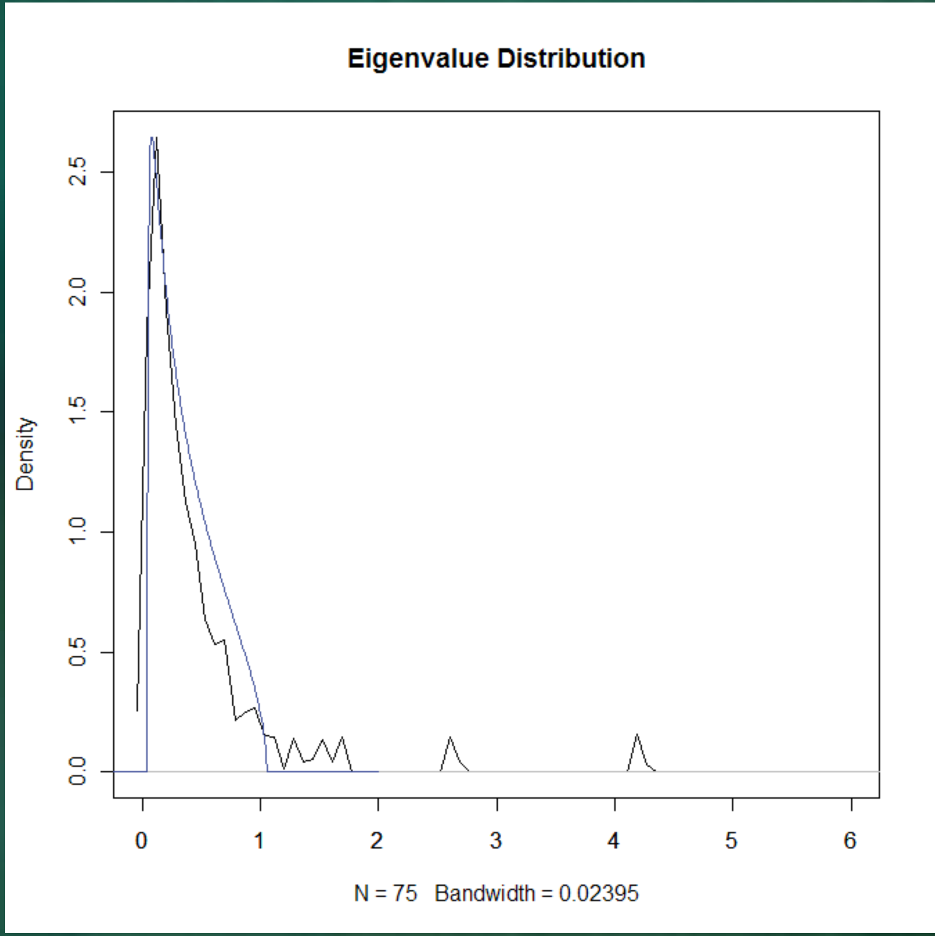


Random Matrix Theory

Fitting the Null Hypothesis

- Daily S&P 500; N=384, T=1200
- Daily S&P 500 subset; N=75, T=200
- Shuffled S&P 500; N=75, T=200

Q = 1.768204
σ = 0.6321195

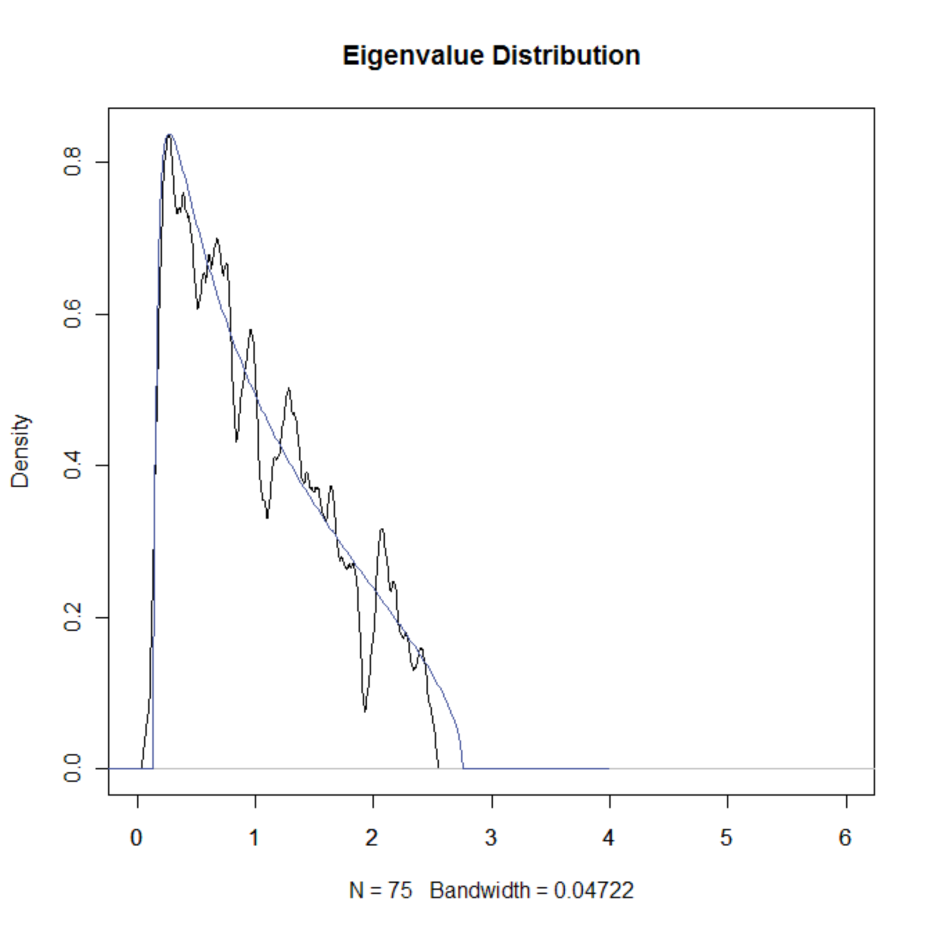


Random Matrix Theory

Fitting the Null Hypothesis

- Daily S&P 500; N=384, T=1200
- Daily S&P 500 subset; N=75, T=200
- Shuffled S&P 500; N=75, T=200

Q = 2.514132
σ = 1.019011



Shrinkage Estimation

- James-Stein revealed that a global mean exists
- Shrinking samples toward a global mean improves accuracy of estimation
- This can be applied to covariance matrices

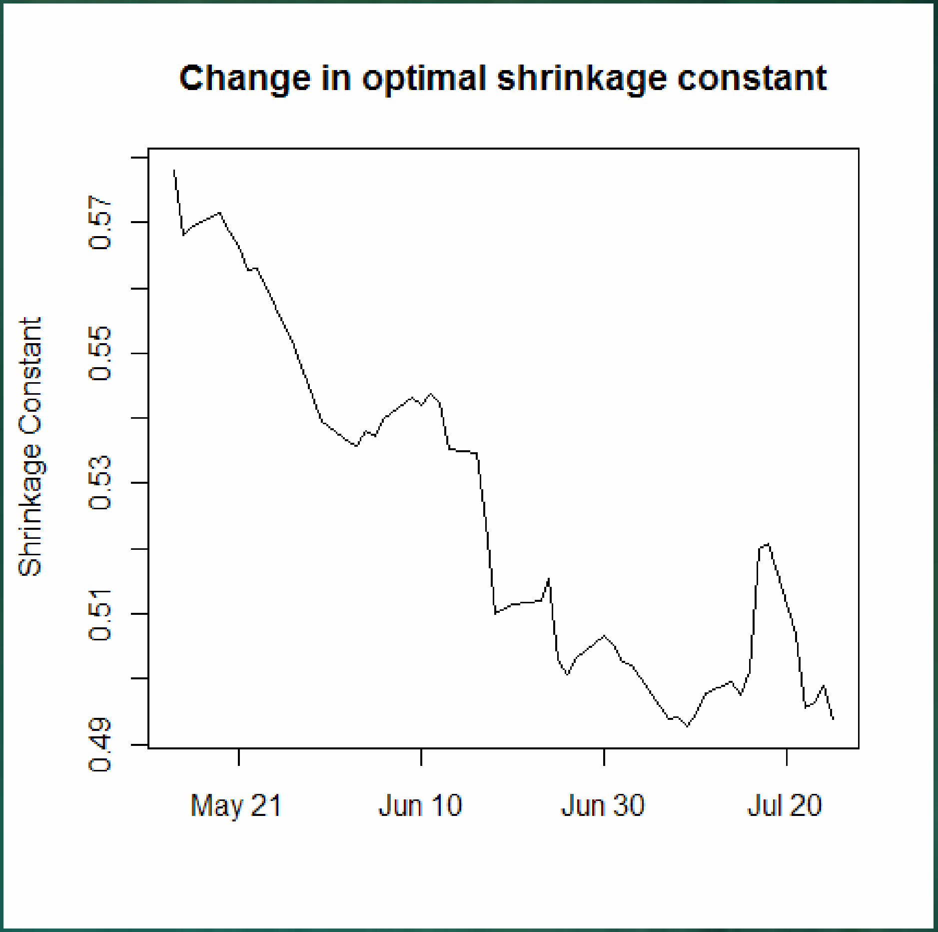
Shrinkage Estimation

What is the global mean?

- The true mean is unknown
- Many candidates exist for covariance
  - Identity matrix
  - Constant correlation matrix
  - Biased estimator (e.g. Barra)

Shrinkage Estimation

Shrinkage Intensity



- Use a single value or calculate per iteration
- Ledoit & Wolf propose optimal coefficient

$$\alpha = \frac{\kappa}{T}$$

$$\kappa = \frac{\pi - \rho}{\gamma}$$

Filtering Correlation Matrices

RMT reconstructs correlation matrix from the empirical correlation matrix by replacing all eigenvalues in noise part of spectrum with their mean

Shrinkage estimation takes a weighted average between the sample covariance and a global mean using a calculated shrinkage constant

Does It Work?

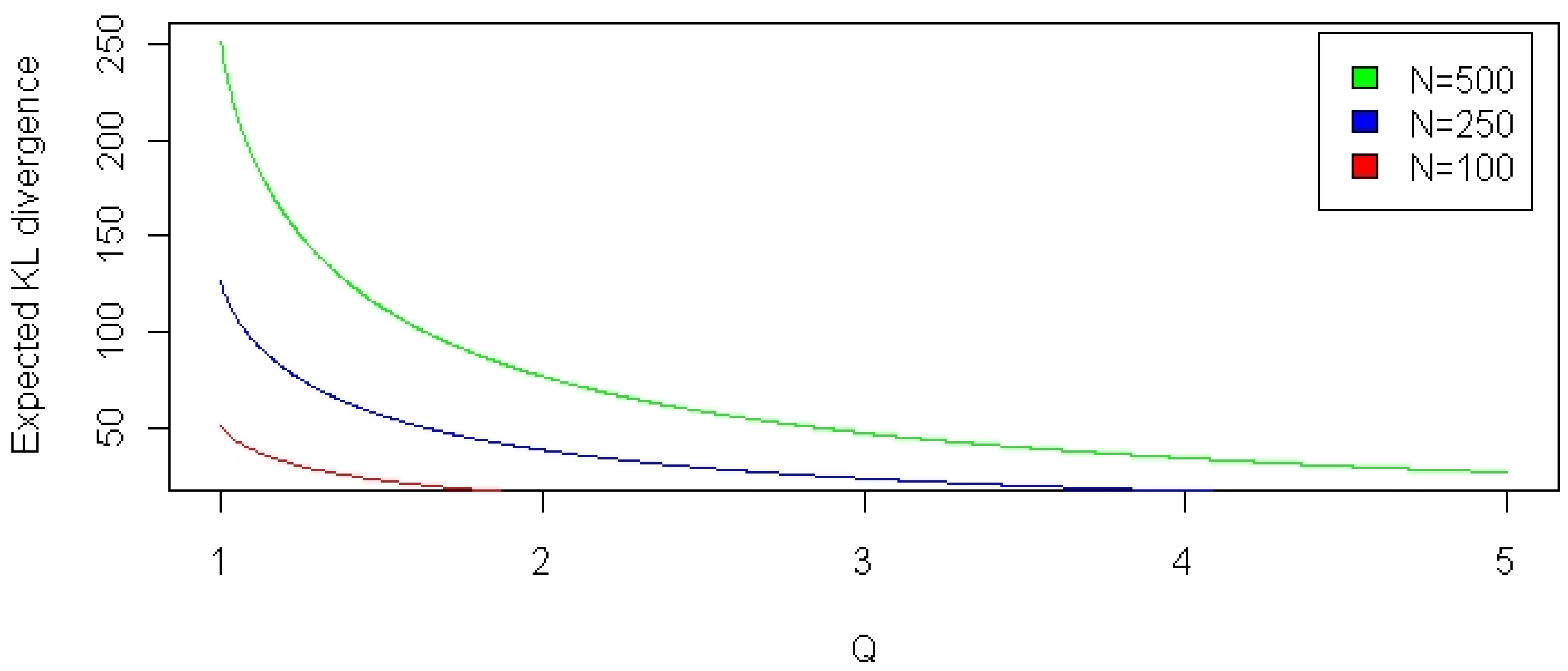
- How do you measure effectiveness?
- Again, two approaches
  - Kullback-Leibler distance
  - Out of sample portfolio returns
- Which will you believe?

Kullback-Leibler Distance

- KL distance measures the entropy between two probability density functions
- Not a true distance - but still useful!
  - Triangle inequality is not satisfied
  - Not symmetric
- Can measure information content and stability

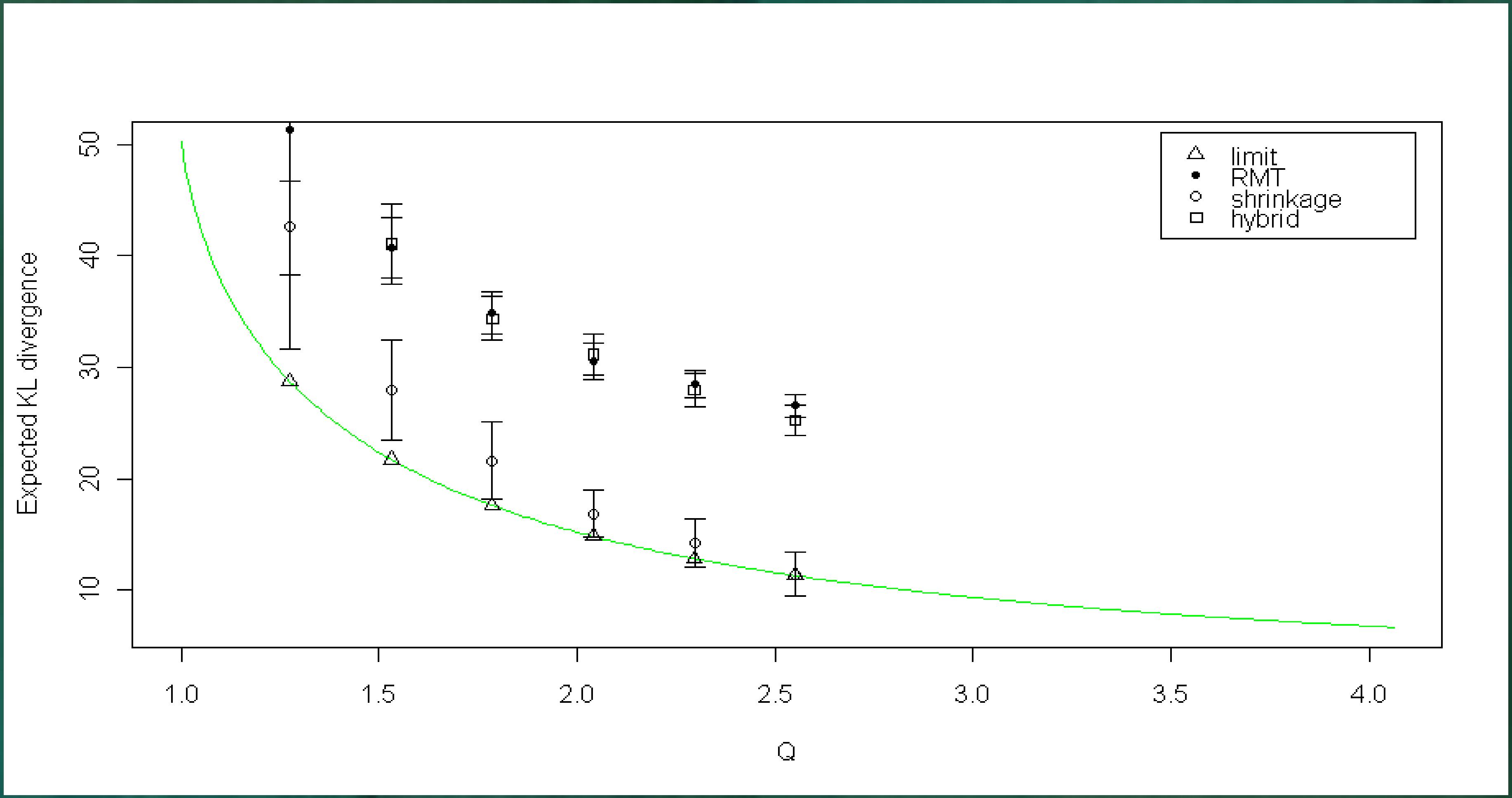
Kullback-Leibler Distance

Theoretical Limit



Kullback-Leibler Distance

Empirical Results



Portfolio Performance

- Minimum variance

SPX random subset (100 assets) – 175 day window, 125 dates

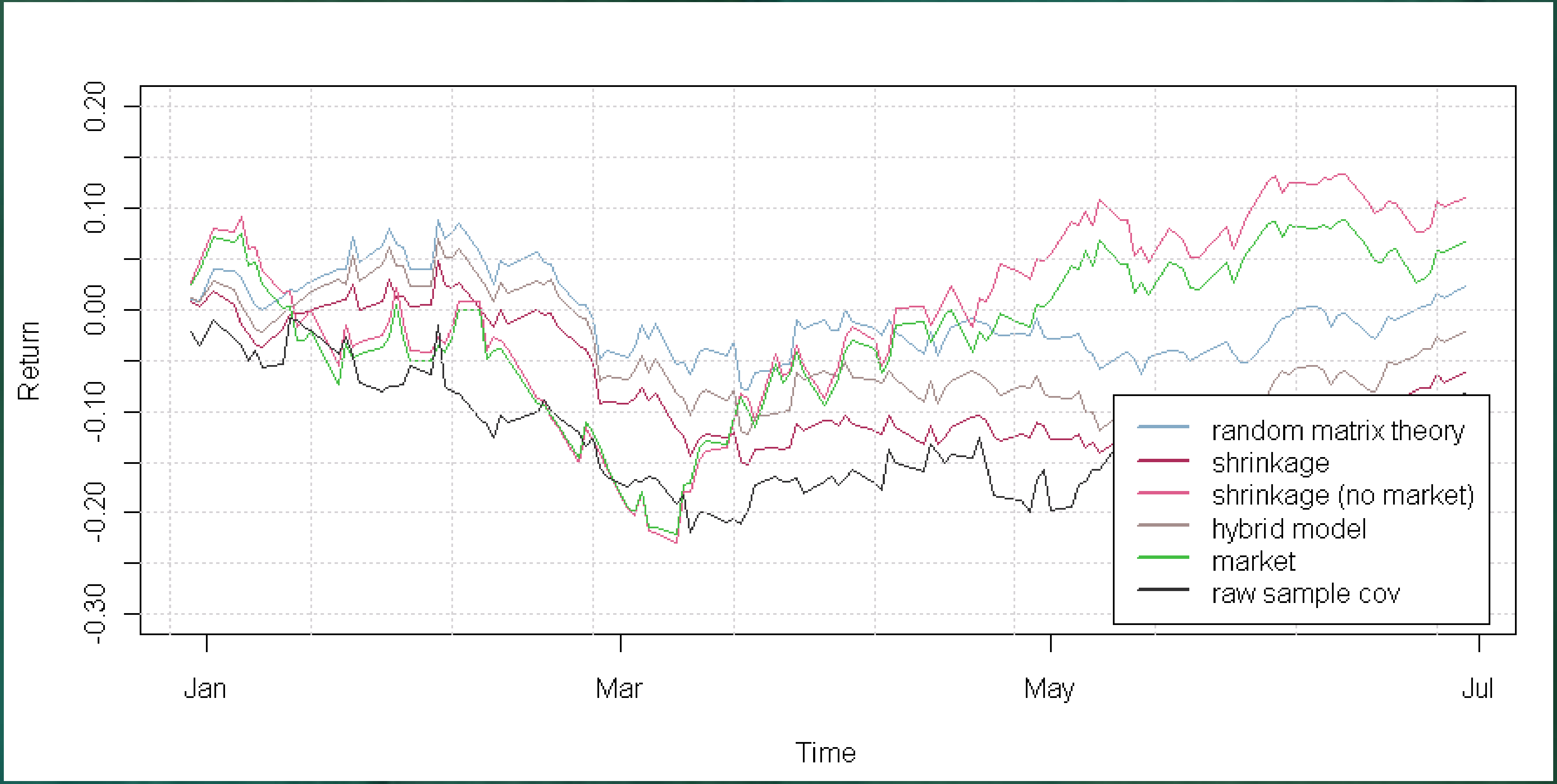
	sharpe.ratio	annual.return	annual.stdev
rmt	0.1911074	0.04646651	0.2431435
shrink	-0.5547973	-0.12035726	0.2169392
shrink.m	0.6403425	0.23386712	0.3652219
hybrid	-0.1934593	-0.04509580	0.2331023
raw.sample	-0.5535997	-0.15960243	0.2882993
market	0.3956911	0.13857861	0.3502192

SPX random subset (100 assets) – 125 day window, 175 dates

	sharpe.ratio	annual.return	annual.stdev
rmt	-0.73633608	-0.20746138	0.2817482
shrink	-0.83450696	-0.24169547	0.2896267
shrink.m	0.09709427	0.04461285	0.4594797
hybrid	-0.69065240	-0.18980906	0.2748257
raw.sample	0.36170223	0.17826057	0.4928379
market	-0.06505888	-0.02908206	0.4470114

Portfolio Performance

Minimum variance optimization

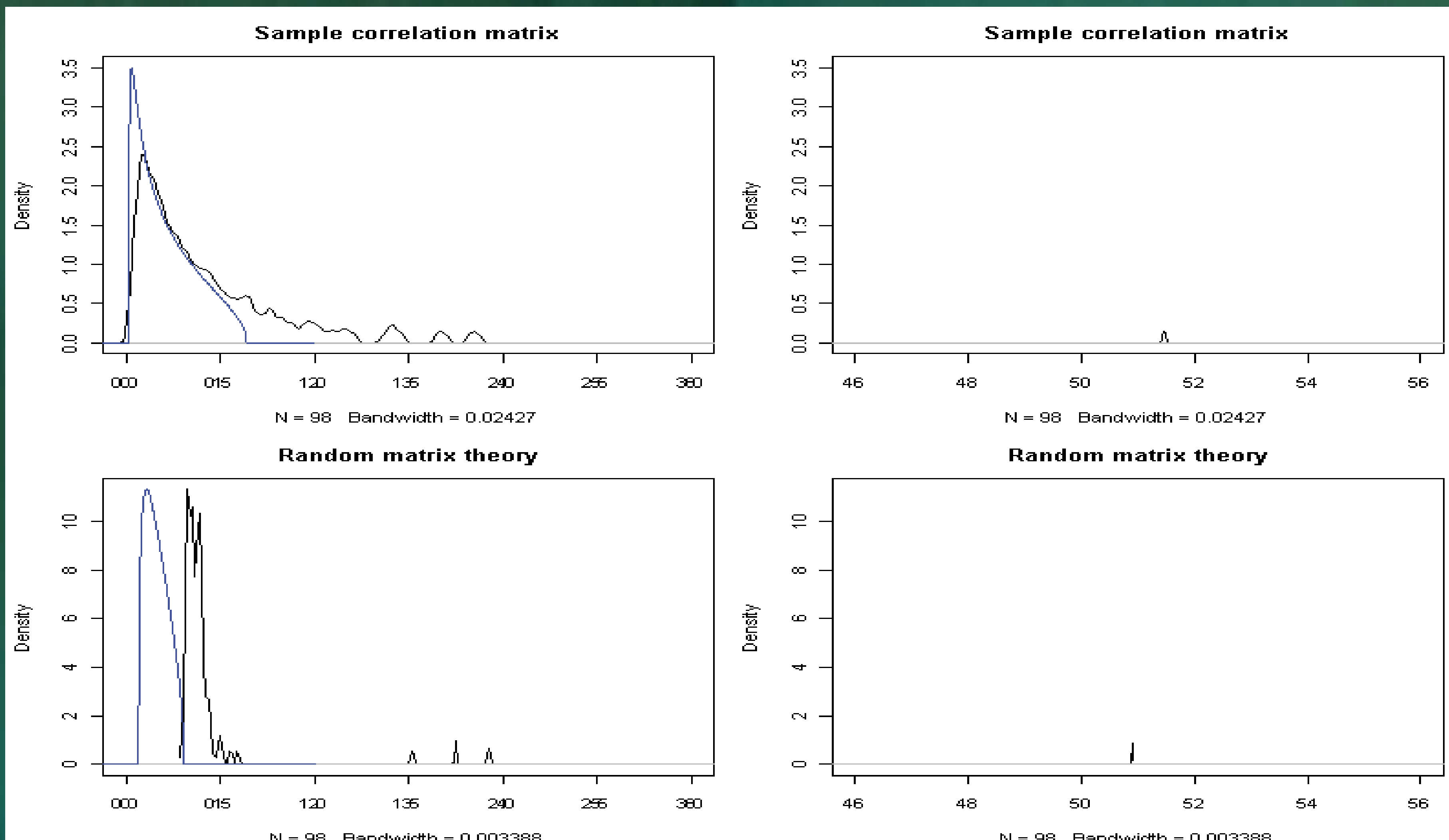


Reconciliation

- Is there a connection between the theories?
- Examine eigenvalue distributions
- What about a hybrid approach?
- What about other eigenvalues?

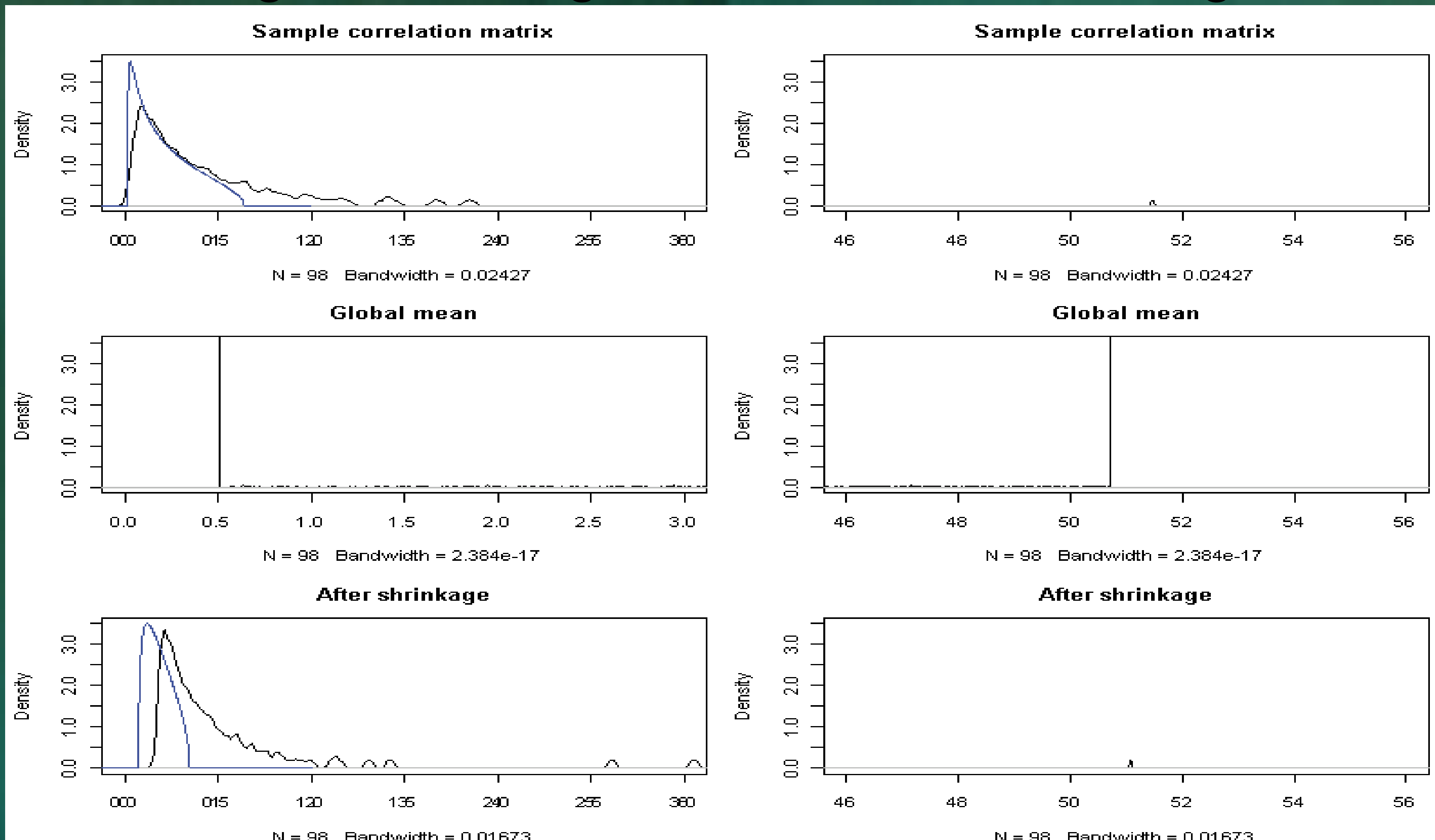
Reconciliation

RMT replaces 'noisy' eigenvalues with average value



Reconciliation

Shrinkage scales eigenvalues towards a single value



Reconciliation

Eigenvalue distributions

- The eigenvalue of the global mean is in the noise part of the RMT spectrum!
- Both methods reduce noise by averaging out noisy eigenvalues
- Difference is in execution
- Hybrid approach has no benefit

References

- Laurent Laloux and Pierre Cizeau and Jean-Philippe Bouchaud and Marc Potters, Random matrix theory and financial correlations, 1999
- M. Potters, J.P. Bouchaud, L. Laloux, Financial Applications of Random Matrix Theory: Old Laces and New Pieces, 2005
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- Olivier Ledoit & Michael Wolf, Honey, I Shrunk the Sample Covariance Matrix, Economics Working Papers 691, Department of Economics and Business, Universitat Pompeu Fabra, 2003
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End

- All images were generated by using Tawny (written by me)
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