A Tale of Two Theories:  
Reconciling random matrix theory and shrinkage estimation as methods for covariance matrix estimation

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Overview

- Motivation
- Random Matrix Theory
- Shrinkage Estimation
- Measuring Effectiveness
  - Kullback-Leibler distance
  - Financial measures
- Reconciliation
Motivation

- Sample covariance $\neq$ true covariance matrix
- Estimation error is large when !(T >> N)
  - Large portfolios
  - Monthly time frame
- Need a good estimate of covariance matrix
Approaches

Physics: Random matrix theory

- Eigenvalue distribution
Approaches

Physics: Random matrix theory

- Eigenvalue distribution
- Null hypothesis
Approaches

Physics: Random matrix theory

- Eigenvalue distribution
- Null hypothesis
- Remove noise component
Approaches

Statistics: Shrinkage Estimation

- Central limit theorem
Approaches

Statistics: Shrinkage Estimation

- Central limit theorem
- Weighted average
  \[ \alpha F + (1-\alpha) S \]
Approaches

Statistics: Shrinkage Estimation

- Central limit theorem
- Weighted average \( \alpha F + (1-\alpha) S \)
- Reduced estimation error
Approaches

Which is Right?
Random Matrix Theory

• Eigenvalue distribution of random matrices is defined by the Marcenko-Pastur limit

\[ \rho(\lambda) = \frac{Q}{2\pi \sigma^2} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda_{min} - \lambda)}}{\lambda} \]

\[ \lambda_{max/min} = \sigma^2 (1 \pm \sqrt{\frac{1}{Q}})^2 \]

• Sample correlation matrices can be filtered to remove this noise

• The reconstructed matrix is then used in portfolio optimization
Random Matrix Theory
Marcenko-Pastur Distributions

- Random matrix with normal distribution; N=1000, T=4000
- Random matrix with normal distribution; N=250, T=1000
- Random matrix with normal distribution; N=50, T=200

![Eigenvalue Distribution](image)
Random Matrix Theory

Marcenko-Pastur Distributions

- Random matrix with normal distribution; $N=1000$, $T=4000$
- Random matrix with normal distribution; $N=250$, $T=1000$
- Random matrix with normal distribution; $N=50$, $T=200$

![Eigenvalue Distribution](image.png)
Random Matrix Theory
Marcenko-Pastur Distributions

- Random matrix with normal distribution; $N=1000, T=4000$
- Random matrix with normal distribution; $N=250, T=1000$
- Random matrix with normal distribution; $N=50, T=200$
Random Matrix Theory
Fitting the Null Hypothesis

- Daily S&P 500; N=384, T=1200
- Daily S&P 500 subset; N=75, T=200
- Shuffled S&P 500; N=75, T=200

\[ Q = 2.072958 \]
\[ \sigma = 0.8152044 \]
Random Matrix Theory

Fitting the Null Hypothesis

- Daily S&P 500; N=384, T=1200
- Daily S&P 500 subset; N=75, T=200
- Shuffled S&P 500; N=75, T=200

\[ Q = 1.768204 \]
\[ \sigma = 0.6321195 \]
Random Matrix Theory

Fitting the Null Hypothesis

- Daily S&P 500; N=384, T=1200
- Daily S&P 500 subset; N=75, T=200
- Shuffled S&P 500; N=75, T=200

$Q = 2.514132$
$\sigma = 1.019011$
Shrinkage Estimation

- James-Stein revealed that a global mean exists
- Shrinking samples toward a global mean improves accuracy of estimation
- This can be applied to covariance matrices
Shrinkage Estimation

What is the global mean?

- The true mean is unknown
- Many candidates exist for covariance
  - Identity matrix
  - Constant correlation matrix
  - Biased estimator (e.g. Barra)
Shrinkage Estimation

Shrinkage Intensity

- Use a single value or calculate per iteration
- Ledoit & Wolf propose optimal coefficient

\[
\alpha = \frac{k}{T} \\
k = \frac{\pi - \rho}{\gamma}
\]
Filtering Correlation Matrices

RMT reconstructs correlation matrix from the empirical correlation matrix by replacing all eigenvalues in noise part of spectrum with their mean.

Shrinkage estimation takes a weighted average between the sample covariance and a global mean using a calculated shrinkage constant.
Does It Work?

- How do you measure effectiveness?
- Again, two approaches
  - Kullback-Leibler distance
  - Out of sample portfolio returns
- Which will you believe?
Kullback-Leibler Distance

- KL distance measures the entropy between two probability density functions
- Not a true distance - but still useful!
  - Triangle inequality is not satisfied
  - Not symmetric
- Can measure information content and stability
Kullback-Leibler Distance

Theoretical Limit

![Graph showing the expected KL divergence vs Q for different N values.](image)
Kullback-Leibler Distance

Empirical Results

![Graph showing empirical results for Kullback-Leibler distance]

- **Expected KL divergence** vs. **Q**
- Different markers represent different methods:
  - Triangle: Limit
  - Circle: RMT
  - Square: Shrinkage
  - Diamond: Hybrid

The graph illustrates the empirical results for various values of Q, with error bars indicating variability.
## Portfolio Performance

- **Minimum variance**

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Portfolio Performance

Minimum variance optimization
Reconciliation

- Is there a connection between the theories?
- Examine eigenvalue distributions
- What about a hybrid approach?
- What about other eigenvalues?
Reconciliation
RMT replaces 'noisy' eigenvalues with average value
Reconciliation

Shrinkage scales eigenvalues towards a single value

Sample correlation matrix

Global mean

After shrinkage
Reconciliation

Eigenvalue distributions

- The eigenvalue of the global mean is in the noise part of the RMT spectrum!
- Both methods reduce noise by averaging out noisy eigenvalues
- Difference is in execution
- Hybrid approach has no benefit
References

- Laurent Laloux and Pierre Cizeau and Jean-Philippe Bouchaud and Marc Potters, Random matrix theory and financial correlations, 1999
End

- All images were generated by using Tawny (written by me)
- Download Tawny from CRAN
- https://nurometic.com
- b_rowe@ml.com or r@nurometic.com