Approximate Conditional-mean Type Filtering for State-space Models

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joint work with

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Linear State Space Models

\* State equation:

\[ x_t = \Phi x_{t-1} + \varepsilon_t \]

\* Observation equation:

\[ y_t = H x_t + v_t \]

\* Ideal model assumptions:

\[ x_0 \sim \mathcal{N}_p(\mu_0, \Sigma_0), \quad \varepsilon_t \sim \mathcal{N}_p(0, Q), \quad v_t \sim \mathcal{N}_q(0, R), \]

all independent
Classical Kalman Filter

- **Initialization** \((t = 0)\):

  \[ x_{0|0} = \mu_0, \quad P_0 = \Sigma_0 \]

- **Prediction** \((t \geq 1)\):

  \[ x_{t|t-1} = \Phi x_{t-1|t-1} \]
  \[ M_t = \Phi P_{t-1} \Phi^\top + Q = \text{Cov}(x_{t|t-1}) \]

- **Correction** \((t \geq 1)\):

  \[ x_{t|t} = x_{t|t-1} + K_t (y_t - H x_{t|t-1}) \]
  \[ P_t = M_t - K_t H M_t = \text{Cov}(x_{t|t}) \]

with \( K_t = M_t H^\top (H M H^\top + R)^{-1} \) (Kalman gain)
Types of Outliers

- Innovational Outliers (IO’s):
  - state equation is contaminated
  - not considered here

- Additive Outliers (AO’s):
  - observations are contaminated
  - error process $v_t$ is affected
  - possible model:

\[
CN_q(\gamma, R, R_c) = (1 - \gamma)N_q(0, R) + \gamma N_q(\mu_c, R_c)
\]

- Other Types of Outliers:
  - substitutive outliers (SO’s)
  - patchy outliers
Masreliez’s Theorem (1975)

If $x_t|Y_{t-1} \sim \mathcal{N}_p(x_{t|t-1}, M_t)$, $t \geq 1$, then

$x_{t|t} = E(x_t|Y_t), t \geq 1$, is generated by the recursions

\[
x_{t|t} = x_{t|t-1} + M_t H^\top \Psi_t(y_t)
\]

\[
P_t = M_t - M_t H^\top \Psi'_t(y_t) H M_t
\]

\[
M_{t+1} = \Phi P_t \Phi^\top + Q,
\]

with $(\Psi_t(y))_i = - (\partial/\partial y_i) \log f_{y_t}(y|Y_{t-1})$ and

$(\Psi'_t(y))_{ij} = (\partial/\partial y_j)(\Psi_t(y))_i$.

**Ψ_t(y)** is called the score function.

**Note:** If $f_{y_t}(.,|Y_{t-1})$ is Gaussian, Masreliez’s filter reduces to the Kalman filter.
The Score Function $\Psi_t$
Multivariate ACM-type Filter

- approximate conditional-mean (ACM) type filter
- proposed by B. Spangl and R. Dutter (2008)
- modified correction step:

\[
\begin{align*}
x_{t|t} &= x_{t|t-1} + M_t H^\top S_t \psi(S_t(y_t - H x_{t|t-1})) \\
P_t &= M_t - M_t H^\top S_t \psi'(S_t(y_t - H x_{t|t-1})) S_t H M_t
\end{align*}
\]

for an \( S_t \) and a \( \psi \)-function appropriately chosen

- in the case univariate observations equivalent to Martin’s ACM type filter (Martin, 1979)
Huber’s Multivariate Psi-function
Hampel’s Multivariate Psi-function

(a) 

(b)
Approximating the Score Function
**rLS Filter**

- proposed by P. Ruckdeschel (2001)
- modified correction step:

\[ x_{t|t} = x_{t|t-1} + H_b(K_t(y_t - Hx_{t|t-1})) \]

with \( H_b(z) = z \min\{1, b/\|z\|_2\} \) and \( \| . \|_2 \) the Euclidean norm

- optimal for SO's in some sense
Simulation

* State Space Process:
  * simulate state space process using two different sets of hyper parameters
  * and AO’s from two different contamination setups:
    \[
    \mathcal{N}_2\left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix} \right) \text{ or } \mathcal{N}_2\left( \begin{pmatrix} 25 \\ 30 \\ 0 \\ 0 \\ 0.9 \end{pmatrix}, \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix} \right).
    \]
  * vary contamination $\gamma$ from 0\% to 20\% by 5\%
  * each 400 times

* Filtering:
  * robust filtering (ACM, rLS)

* Evaluation:
  * compare with true state process via MSE
Simulation (cont.)

Example I:

\[
\begin{align*}
\mu_0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \Phi &= \begin{pmatrix} 0.5 & 0.3 \\ 0.6 & 0.5 \end{pmatrix}, & Q &= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \\
\Sigma_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & H &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & R &= \begin{pmatrix} 2 & -0.2 \\ -0.2 & 0.5 \end{pmatrix}.
\end{align*}
\]

Example II:

\[
\begin{align*}
\mu_0 &= \begin{pmatrix} 20 \\ 0 \end{pmatrix}, & \Phi &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & Q &= \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix}, \\
\Sigma_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & H &= \begin{pmatrix} 0.3 & 1 \\ -0.3 & 1 \end{pmatrix}, & R &= \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.
\end{align*}
\]
Results

1. coordinate of state process

- contamination level 0%
- contamination level 10%
- contamination level 20%

2. coordinate of state process

- contamination level 0%
- contamination level 10%
- contamination level 20%
Results (cont.)

contamination level 0%

contamination level 10%

contamination level 20%

1. coordinate of state process

true
class.
rLS
ACM

2. coordinate of state process

true
class.
rLS
ACM

B. Spangl et al., Approximate Conditional-mean Type Filtering for State-space Models – p. 16/20
The R package robKalman

- general function `recursiveFilter` with parameters:
  - observations
  - state-space model (hyper parameters)
  - functions for the init./pred./corr. step

- available filters:
  - KalmanFilter, rLSFilter,
    ACMfilter, mACMfilter
  - all: wrappers to `recursiveFilter`
Remarks & Outlook

- ACM performs better than rLS for both contamination situations
- rLS yields larger errors in the case of 0% contamination because it was calibrated to a loss of efficiency $\delta = 10\%$
- all simulations were made with R
- R-package robKalman for filtering already exists (but is still under construction!)
  http://r-forge.r-project.org/projects/robkalman/

- S4 classes for state-space models and filtering results


