Calibrating the evidence in experiments with applications to meta-analysis

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Calibrating the p-value

How much evidence is there in a p-value of 0.01, say, relative to 0.05?

How small must a p-value be to represent twice as much evidence against the null hypothesis as 0.05?

Calibration of the p-value

Given $X = \mu + Z$ we want to test

$\mu = 0$ against $\mu > 0$.

Observe $X = x$; then $PV(x) = \Phi(-x)$.

Under alternatives,

$PV(X) = 1 - \Phi(X)$,

where $X \sim N(\mu, 1)$.

Remarks:

- There are two p-values.
- ‘Evidence’ for the alternative $\mu > 0$, however it is defined, should grow at rate $\sqrt{n}$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.0005</th>
<th>0.001</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T(p)$</td>
<td>3.291</td>
<td>3.090</td>
<td>2.326</td>
<td>1.645</td>
<td>1.276</td>
<td>0.8416</td>
</tr>
<tr>
<td>$T(0.05)$</td>
<td>2.000</td>
<td>1.879</td>
<td>1.414</td>
<td>1.000</td>
<td>0.779</td>
<td>0.511</td>
</tr>
</tbody>
</table>

Now suppose the experimenter makes $n$ measurements $x_1, \ldots, x_n$ and judges the null hypothesis using the average $\bar{x}_n = (x_1 + \cdots + x_n)/n$.

The random p-value based on these $n$ observations can be written $PV_n = 1 - \Phi(\sqrt{n}\bar{X}_n)$.

It follows that the transformed p-value $T(PV_n) = \sqrt{n}\bar{X}_n$ has an expected value $\sqrt{n}\mu$ which is proportional to the square root of the sample size.

A p-value of 0.05 should be reported as evidence $1.645 \pm 1$. 
To test $\theta = 0$ versus $\theta > 0$, let $S$ be a test statistic which rejects $H_0$ for large values of $S$. A measure of evidence $T$ should satisfy:

$E_1$. $T$ is monotone increasing in $S$;

$E_2$. the distribution of $T$ is normally distributed for all values of the parameters;

$E_3$. the variance $\text{Var}[T] = 1$ for all values of the parameters; and

$E_4$. the expected evidence

$$\tau = \tau(\theta) = E_{\theta}[T]$$

is increasing in $\theta$ from $\tau(0) = 0$.

How generally applicable is the calibration scale?

For one-sample $t$-tests, use

$$\sqrt{2\nu} \sinh^{-1}\left(\frac{t\nu}{\sqrt{2\nu}}\right)$$

For one-sample Binomial tests, use

$$2\sqrt{n} \left\{ \arcsin\left(\sqrt{\hat{p}}\right) - \arcsin\left(\sqrt{p_0}\right) \right\}$$

For Chi-squared tests with $X \sim \chi^2_{\nu}(\lambda)$, use

$$\left\{ X - \nu/2 \right\}^{1/2} - \nu/2^{1/2}$$

Combining evidence:

Given $K$ studies measuring possibly different effects $\theta_k$ with evidence for $\theta_k > 0$ given by

$$T_k \sim N(\tau_k, 1),$$

and $\tau_k = \sqrt{n_k} m(\theta_k)$.

How one combines evidence in $(T_1, \ldots, T_K)$ depends on:

1. how much evidence $T_Q$ one finds for heterogeneity of the $\theta_k$’s and

2. on the specific alternative to the joint null $\theta_1 = \ldots = \theta_K$ one wants evidence for.

The main advantage is that it is like doing meta-analysis with known weights.

Summary

- The evidence in the p-value is on the probit scale
- VST’s will put many problems on the probit scale
- Interpreting evidence on the probit scale is simple
- Combining evidence on the probit scale is simple