Graphical Exploratory Analysis Using Halfspace Depth

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Bivariate halfspace depth (Tukey depth)

Take a fixed collection of datapoints:

\((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\).

Given an arbitrary point \((x, y)\):
- take all (closed) halfspaces having \((x, y)\) on their boundary;
- count how many datapoints lie inside them;
- take the minimum of this count over the halfspaces.

That is: the bivariate halfspace depth of a point \(\vartheta = (x, y)\) is the minimal number of the datapoints lying in a closed halfspace containing \(\vartheta\) (on its boundary).

\[
D(\vartheta) = \inf_{u \neq 0} \# \{i : u^T (z_i - \vartheta) \geq 0\},
\]

where \(z_i = (x_i, y_i)\), \(\vartheta = (x, y)\), and \(\# \{\cdot\} = \text{card}\{\cdot\} \).
**Depth = 2 (movie)**

**Tukey depth contours**

Depth contour of level $k \equiv$ set of points with depth $\geq k$. Nested, convex,...

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**Bagplot**

Rousseeuw, Ruts, and Tukey (1999): a bivariate boxplot

Bag: depth contour containing about $1/2$ of observations

Tukey median: a point selected from the contour with maximal depth (various methods possible, the Steiner point is our choice)

Fence: magnified bag (by fudge factor 3, with Tukey median as center)

Outliers: datapoints outside the fence

Loop: the convex hull of the datapoints inside the fence

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**Bagplot in action**

> library(depth)
> bagplot(x,y)
Student depth (location-scale)

Rousseeuw and Hubert (1998), Mizera (2002).
Mizera and Müller (2004): halfspace depth in the Lobachevski geometry of the location-scale space (a shortest, but perhaps not the most understandable definition).

Depth = 2 (movie)

> plot(lsdc(rnorm(100000),'dozen'),maxline=F)
> plot(lsdc(rt(100000,1),'dozen'),maxline=F)

Student depth contours

> plot(lsdc(rivers,"six",maxline = T),paint=terrain.colors(6))
> points(rivers,rivers*0,pch=16)

Computer science

In general, NP hard. But plotting fortunately only dim 2.
Student depth contours: $O(n)$, apart from the initial $O(n \log n)$ sorting.
Tukey depth: all contours $O(n^2)$ (but who needs them all?)
Individual depth contours: better? Yes - at least in theory...
Practical algorithm (jointly with David Eppstein): a dynamic convex hull structure (updating strategy).
Implementation: R / ... ?
Interpreted languages (Matlab, R, Python, Lisp) are fun ... but slow. Compiled languages (machine code, assembly, FORTRAN, C(++) , Java) are fast...
... but are work (= no fun).
A case study of useR psychoanalysis ($n = 1$)

- FORTRAN avoided (trauma from childhood).
- C routines running (translated from MATLAB, a labor therapy).
- Python prototypes of my co-author David Eppstein deciphered (still waking up at night).
- Segmentation fault for $n > 100000$ taken care of (thanks to Duncan Temple Lang for the $S_{\text{alloc}}$ command!)
- The next use of $S_{\text{alloc}}$ command successfully guessed (without finding any documentation or asking DTL once again).
- Poor Man's Zoom - a Wittgensteinian approach to graphics.
- Eventually, learned how to pass R CMD check (man gets accustomed even to gallows, a Slovak proverb).
- And never ever asked anything on R-help.
- It's almost done. (By the anniversary of October revolution?)

Frustrations of a random sample unit: in the search of identity

- (Pressburger blut or Midwesterner in a broad sense?.)
- Computational statistician? Oh, no FORTRAN, thanks...
- UseR from 1998? Bring two witnesses, please. (UseR < 2000 $\approx$ NSDAP < 1933 or Czechoslovak Communist Party < 1948)
- Besides, useRs don’t worry about things like segmentation faults and $S_{\text{alloc}}$ documentation.
- DevelopeR then? Oh, don’t make me blushing...
- AbuseR. Self-promotion, albeit with attacks of guilty feelings (will a confession get me a pardon?).
- “Don’t work on software, work on ideas” (Rich Sutton, a computer science Zen Master from Edmonton).

Warning

ALTHOUGH ABUSING R WAS NOT PROVED TO BE ADDICTIVE, IT SHOULD BE NOTED THAT IT OFTEN LEADS TO HARDER STUFF.
Viennese epilogue

Stefan Zweig

Theodor Herzl

Some ideas carry a lot of power...

...and the genie is out of the bottle.

Also:

“That what is, often prevails over what could, or even over what should be.”

Is it Fellini? (A reward offered for help with this.)