Basic principles of bond pricing

- **yield to maturity**
  \[ p_c + a = C \sum_{i=1}^{n} e^{-y m_i} + R e^{-y m_n} \]

- equivalent formulation of the bond price equation uses the discount factors \( d_i = \delta(m_i) = e^{-s m_i} \)
- continuous discount function \( \delta(\cdot) \) is formed by interpolation of the discount factors
  \[ p_c + a = C \sum_{i=1}^{n} \delta(m_i) + \delta(m_n)R \]

- implied \( j \)-period forward rate
  \[ f_{ij} = \frac{j s_j - t s_t}{j - t} \]

- duration is a weighted average of time to cash flows
  \[ D = \frac{1}{p_c + a} \left[ C \sum_{i=1}^{n} \delta(m_i)m_i + \delta(m_n)Rm_n \right] \]

Term structure estimation

- estimate zero-coupon yield curves and credit spread curves from market data
- usual way for calculation of credit spread curves
  \[ c_i(t) = s_i(t) - s_{ref}(t) \]
- parsimonious approach widely used by central banks

<table>
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<tr>
<th>Maturities</th>
<th>Yields</th>
<th>Spread curves</th>
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</table>

GERMANY
AUSTRIA
ITALY
**Instantaneous forward rates**

\[ f(m, b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_2}\right) \]

**Spot rates**

\[ s(m, b) = \beta_0 + \beta_1 \frac{1 - \exp(-m/\tau_1)}{m/\tau_1} + \beta_2 \left( \frac{1 - \exp(-m/\tau_1)}{m/\tau_1} - \exp(-m/\tau_1) \right) \]

**Objective function**

\[ b_{opt} = \min_b \sum_{i=1}^n \omega_i (\hat{P}_i - P_i)^2 \quad \text{weighted price errors} \]

\[ b_{opt} = \min_b \sum_{i=1}^n (\hat{y}_i - y_i)^2 \quad \text{yield errors} \]

**Extensions**

- Svensson (1994) extended the functional form by two additional parameters which allows for a second hump-shape

\[ f(m, b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_2}\right) + \beta_4 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right) \]

- simple calculation method of credit spread curves could lead to twisting curves

- Jankowitsch and Pichler (2004) proposed a joint estimation method, which leads to smoother and more realistic credit spread curves

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