Robust Estimation for Circular Data using R

Claudio Agostinelli
claudio@unive.it

Dipartimento di Statistica
Università Ca' Foscari di Venezia
San Giobbe, Cannaregio 873, Venezia
Tel. 041 2347446, Fax. 041 2347444
http://www.dst.unive.it/~claudio

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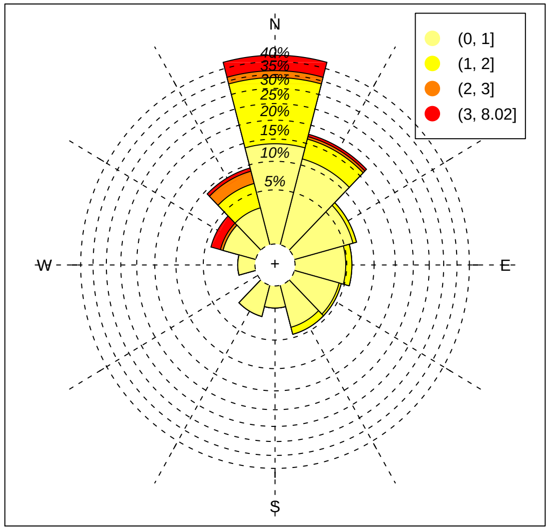
Outline

Circular Data

Circular data arises in many diverse scientific fields like in Natural, Physical, Medical and Social Sciences where some or all the measurements are directions. The two main ways correspond to the two principal circular measuring instruments:

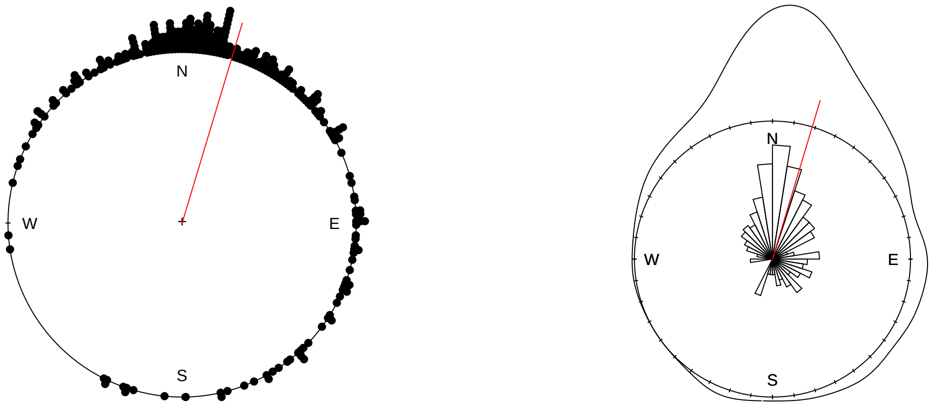
- the *compass*
  - e.g. wind directions and directions of migrating animals.
- the *clock*.
  - e.g. arrival times (on a 24-hour clock) of subjects.

Because of the nature of the circular observations, the analysis (descriptive and/or inferential) cannot be carried out with standard methods for observations on Euclidean space.
Some references: Mardia and Jupp [2000], Jammalamadaka and SenGupta [2001] and Jupp and Mardia [1989].



Wind rose of the dataset from Col de la Roa (Italy) meteorological station in March and April 2001 between 3.00 and 4.00 o'clock (n=310).

R code



Stack Circular Plot (Left) and estimated density (Right, together with a Rose Diagram) of the Wind direction dataset. ‣ R code

Models for Circular data

Some examples are:

- Von Mises (Normal Circular) distribution:

$$m(x;\mu,\kappa) = \frac{1}{2\pi I_0(x)} \exp(\kappa cos(x-\mu)), \qquad \kappa \geq 0$$

- Wrapped Normal distribution:

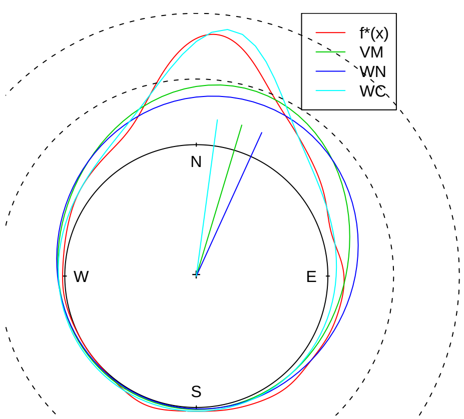
$$m(x;\mu,\rho) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty} \rho^{p^2} \cos p(x-\mu)\right), \qquad \rho = \exp\left(\frac{-\sigma^2}{2}\right)$$

- Wrapped Cauchy distribution:

$$m(x;\mu,\rho) = \frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho cos(x-\mu)}, \qquad \rho = \exp(-\sigma)$$

- Cardioid distribution:

$$m(x;\mu,\rho) = \frac{1+2\rho cos(x-\mu)}{2\pi}, \qquad -\frac{1}{2} < \rho < \frac{1}{2}$$



MLE estimators for the Wind direction dataset using

- Von Mises (VM, $\hat{\mu} = 16.74$, $\hat{\kappa} = 1.76$);
- Wrapped Normal (WN, $\hat{\mu} = 24.487$, $\hat{\rho} = 0.603$);
- Wrapped Cauchy (WC, $\hat{\mu} = 7.662$, $\hat{\rho} = 0.697$). ‣ R code.

Robust Statistics

Robust Statistics is a discipline which deals with the inference problem in the case of misspecification of the model with respect to the distribution of the data. A "robust" estimator is able to well describe the parameters, or some features, of the "majority" (or "bulk") of the data. And to quote:

- Hampel et al. [1986], pag. 56: "whereas in classical statistics the model has to fit all the data, in robust statistics it *may* be enough that it fit the majority of the data, the remainder being regarded as outliers";

Some references: Collett [1980], Lenth [1981], Upton [1993] and SenGupta and Laha [2001]. A survey is He [1992].

Outliers in Circular data

- In Euclidean setting the outliers are defined as:
  - Barnett et al. [1994], pag. 7: "... an outliers in a set of data to be an observation (or subset of observations) which appears to be inconsistent with the remainder of that set of data";
  - Hawkins [1980]: an outlier is "an observation that deviates so much from other observations as to arouse suspicions that it was generated by a different mechanism";

  The definition is based on some **"geometric" distance** between observations.
- In Circular setting: the sample space is bounded and the parametric space is often bounded too (e.g. the parametric space of the mean direction in the Von Mises distribution is every interval of length 2π, in general, $[0, 2\pi)$).

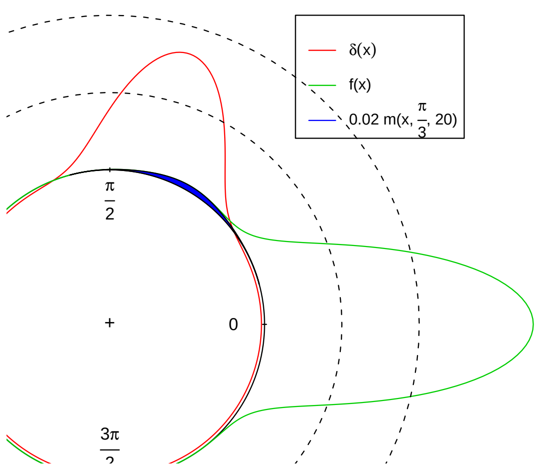
Outliers in Circular Data: Pearson Residuals

In our approach, outliers are observations that are highly unlikely to occur under the assumed model [see Markatou et al., 1995].

This definition is well adapted in the circle since it is based on a **"probabilistic" distance**. One way to measure this discrepancy is to use the Pearson Residuals [Lindsay, 1994] defined as follows

$$\delta(x, \theta, f^*) = \frac{f^*(x)}{m^*(x;\theta)} - 1$$

where $f^*(x)$ is a non parametric density estimator based on the data and $m^*(x;\theta)$ is a smoothed version of the density of the model. Note that $\delta(x, \theta, f^*) = \delta(x \mod (2\pi), \theta, f^*)$.



Pearson Residuals ($\delta(x; \theta = (0, 5), f(x))$) for $f(x) = 0.98m(x; 0, 5) + 0.02m(x; \pi/3, 20)$ where $m(x; \mu, \kappa)$ is the density of a Von Mises distribution. R code

Minimum Distance Estimators

For continuous models the Power Divergence Measure [Cressie and Read, 1988] between the densities $f^*(x)$, $m^*(x, \theta)$ is

$$\int_0^{2\pi} \frac{\left(\frac{f^*(x)}{m(x;\theta)}\right)^{\alpha+1}}{\alpha\ (\alpha+1)}\ m(x;\theta)\ dx$$

or using the Pearson Residual function (in the unsmoothed model) $\delta(x; \theta, f^*)$:

$$\int_0^{2\pi} G(\delta(x;\theta, f^*)) m(x;\theta)\ dx$$

where $G(\delta(x;\theta, f^*)) = \frac{(\delta(x;\theta,f^*)+1)^{\alpha+1}}{\alpha\ (\alpha+1)}$.
Examples are:

- $\alpha = -1/2$: Hellinger distance;
- $\alpha \to -1$: Kullback–Leibler divergence;
- $\alpha = -2$: Neyman's Chi–Square.

Numerical Calculation of Distances

Since $f^*(x)$ and $m^*(x;\theta)$ and their transformations are 2π periodic functions the Power Divergence Measure is computed easily and fast by Fast Fourier Transform and Parseval's formula. In fact, let $\phi_k(\cdot)$ the Fourier transform, we get

$$2\pi \int_0^{2\pi} f^*(x)^{\alpha+1}\, m^*(x;\theta)^{-\alpha}\,\mathrm{d}x = \sum_{k=-\infty}^{\infty} \phi_k\left(f^*(x)^{\alpha+1}\right)\phi_k\left(m^*(x;\theta)^{-\alpha}\right)$$

Weighted Likelihood Estimating Equations

The estimating equations of WLEE is a modified version of the MLE equations where at each score is associated a weight defined as follows

$$w(x;\theta,f^*) = \frac{A(\delta(x;\theta,f^*))+1}{\delta(x;\theta,f^*)+1}$$

where $A(\delta)$ is the Residual Adjustment Function, with the form related to the Power Divergence Measure given by

$$A(\delta) = \frac{(\delta+1)^{\alpha+1}-1}{\alpha+1}$$

Hence the WLEE estimator is the solution of

$$\sum_{i=1}^{n} w(x_i;\theta,f^*)u(x_i;\theta) = 0$$

where $u(x;\theta)$ is the score function for the model.

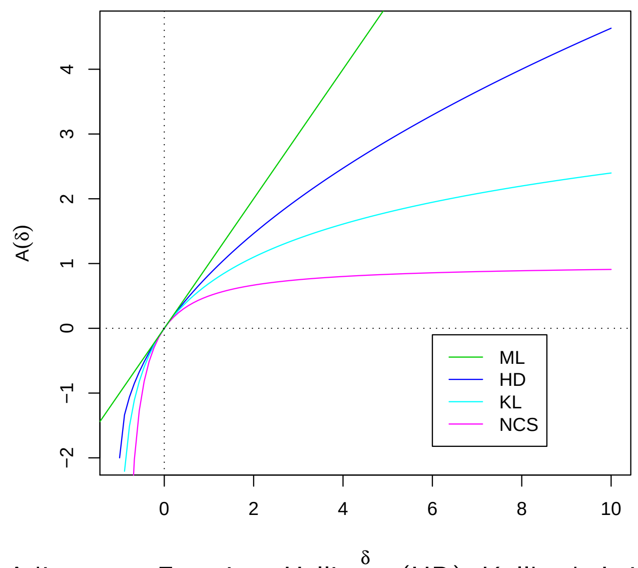
Example: Von Mises distribution

The WLEE for the Von Mises distribution is

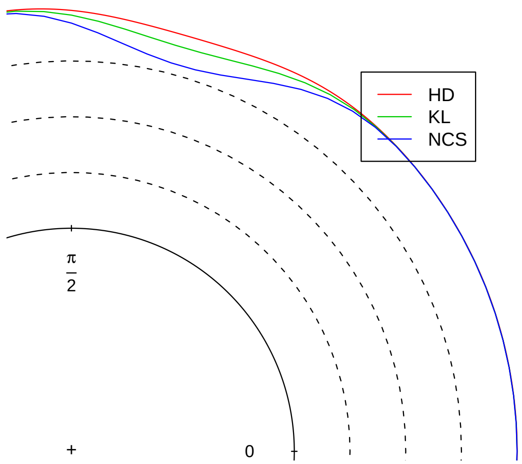
$$\begin{cases} \mu = \arctan^*\left(\frac{\sum_{i=1}^n w(x_i;\mu,\kappa)\ \sin(x_i)}{\sum_{i=1}^n w(x_i;\mu,\kappa)\ \cos(x_i)}\right) \\ \kappa = \mathrm{A}^{-1}\left(\frac{\sum_{i=1}^n w(x_i;\mu,\kappa)\ \cos(x_i-\mu)}{\sum_{i=1}^n w(x_i;\mu,\kappa)}\right) \end{cases}$$

where

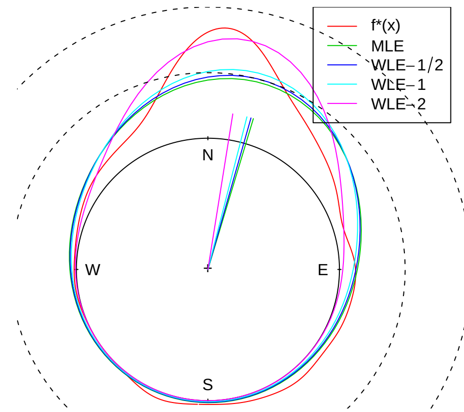
- $\mathrm{A}(\kappa) = \mathrm{I}_1(\kappa)/\mathrm{I}_0(\kappa)$ is the ratio of the two modified Bessel functions of the first kind with order zero and one;
- $\arctan^*$ is the "quadratic-specific" inverse of the tangent that provides the unique inverse of the tangent in $[0, 2\pi)$.



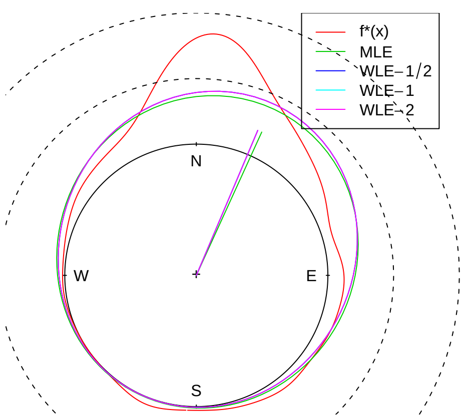
Residual Adjustment Function: Hellinger (HD), Kullback–Leibler (KL), Neyman's Chi–square (NCS) ▸ R code



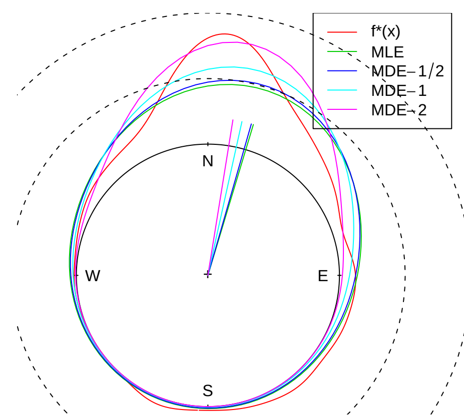
Weight function $(w(x; \theta = (0, 5), f(x)))$ for $f(x) = 0.98m(x; 0, 5) + 0.02m(x; \pi/3, 20)$ where $m(x; \mu, \kappa)$ is the density of a Von Mises distribution. For Residual Adjustment Function: Hellinger (HD), Kullback–Leibler (KL), Neyman's Chi–square (NCS) R code



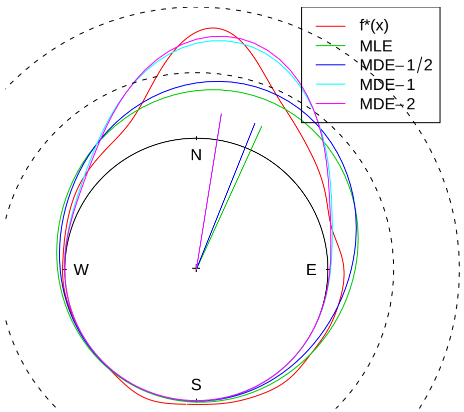
WLE estimators for the Wind direction dataset using the Von Mises model. R code



WLE estimators for the Wind direction dataset using the Wrapped Normal model. R code



MDE estimators for the Wind direction dataset using the Von Mises model. R code



MDE estimators for the Wind direction dataset using the Wrapped Normal model. ▸ R code

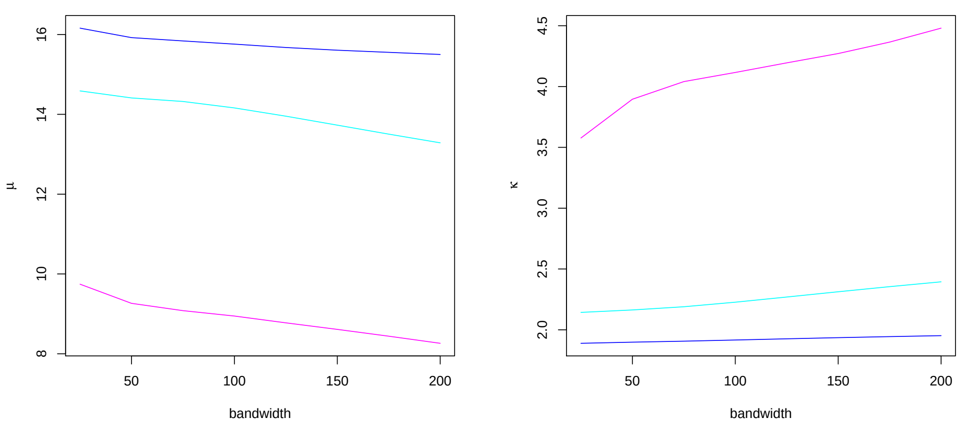
Results for the Wind dataset

Von Mises Model

	MLE	MDE -1/2	MDE -1	MDE -2	WLE -1/2	WLE -1	WLE -2
μ	16.740	15.966	12.456	9.124	15.840	14.322	9.081
κ	1.760	1.961	2.631	4.228	1.907	2.190	4.041

Wrapped Normal Model

	MLE	MDE -1/2	MDE -1	MDE -2	WLE -1/2	WLE -1	WLE -2
μ	24.487	21.787	9.156	9.084	22.414	22.414	22.414
σ	1.005	0.859	0.526	0.504	0.907	0.907	0.907



WLE estimators for the Wind direction dataset using the Von Mises model and different bandwidth. ▸ R code

Conclusions

- We introduced Robust estimators for circular data based on
  - Minimum Distance
  - Weighted Likelihood
- We used, mainly, packages `circular` and `wle`;
- The `Sweave` of the presentation and the dataset would be availables at: `www.dst.unive.it/~claudio/R/index.html`;
- useR! Focus Session on Robust Statistics is Friday, 3:00pm at HS 0.7!

```
> library(circular)
> dataset <- read.table(file = "./wind.dat",
+     sep = ";", header = TRUE)
> mag <- dataset$mag
> dir <- dataset$dir
> ok <- complete.cases(dir, mag)
> mag <- mag[ok]
> magt <- mag
> magt[magt > 3] <- 3.05
> dir <- circular(dir[ok], units = "degrees",
+     template = "geographics")
```



```
> breaks <- circular(0:11/6 * pi - pi/12)
> windrose(x = dir, y = magt, increment = 1,
+     breaks = breaks, fill.col = rev(heat.colors(4)),
+     label.freq = TRUE, main = "")
> legend(x = 0.7, y = 1.2, legend = c("(0, 1]",
+     "(1, 2]", "(2, 3]", "(3, 8.02]"), pch = 19,
+     col = rev(heat.colors(4)), pt.cex = 2)
```

Return


```
> plot(dir, stack = TRUE, ylim = c(-1, 1.22),
+     frame = TRUE)
> mc <- mean(dir)
> lines(c(mc, mc), c(-1, 0.2), col = 2)
> plot(density(dir, bw = 40), ticks = FALSE,
+     ylim = c(-1, 1.8), main = "", xlab = "",
+     ylab = "")
> rose.diag(dir, prop = 2, bins = 40, add = TRUE)
> lines(c(mc, mc), c(-1, 0.2), col = 2)
```

Return


```
> mvm <- mle.vonmises(dir)
> mwn <- mle.wrappednormal(dir)
> mwc <- mle.wrappedcauchy(dir)
> plot(density(dir, bw = 40), ticks = FALSE,
+     xlim = c(-0.8, 1.4), ylim = c(-1, 1.94),
+     main = "", xlab = "", ylab = "", col = 2)
> plot.function.circular(function(x) dvonmises(x,
+     mu = mvm$mu, kappa = mvm$kappa), join = TRUE,
+     add = TRUE, col = 3)
> plot.function.circular(function(x) dwrappednormal(x,
+     mu = mwn$mu, rho = mwn$rho), join = TRUE,
+     add = TRUE, col = 4)
> plot.function.circular(function(x) dwrappedcauchy(x,
+     mu = mwc$mu, rho = mwc$rho), join = TRUE,
+     add = TRUE, col = 5)
```

```
> lines(c(mvm$mu, mvm$mu), c(-1, 0.2), col = 3)
> lines(c(mwn$mu, mwn$mu), c(-1, 0.2), col = 4)
> lines(c(mwc$mu, mwc$mu), c(-1, 0.2), col = 5)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.5, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(1, 100), lty = 2, col = 1)
> legend(0.8, 2, legend = c("f*(x)", "VM", "WN",
+     "WC"), col = 2:5, lty = rep(1, 4))
```

Return


```
> delta <- function(x, eps = 0.02) {
+     res <- ((1 - eps) * dvonmises(x, circular(0),
+         5) + eps * dvonmises(x, circular(pi/3),
+         20))/dvonmises(x, circular(0), 5) -
+         1
+     return(res)
+ }
> curve.circular(delta, n = 501, join = TRUE,
+     xlim = c(-0.6, 2.7), ylim = c(-0.6, 1.8),
+     col = 2, tcl.text = 0.2, cex = 0.9)
> plot.function.circular(function(x) 0.98 * dvonmises(x,
+     circular(0), 20) + 0.02 * dvonmises(x,
+     circular(pi/3), 20), n = 501, join = TRUE,
+     add = TRUE, col = 3)
> lines(circular(seq(0, 2 * pi, length = 100)),
```



```
+     rep(0.5, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(1, 100), lty = 2, col = 1)
> legend(1.2, 2, legend = c(expression(delta(x)),
+     "f(x)", expression(paste("0.02 m(x, ",
+         frac(pi, 3), ", 20)", sep = ""))),
+     lty = rep(1, 3), col = c(2, 3, 4), cex = 0.8)
> cont <- function(x) {
+     den <- 0.02 * dvonmises(x, circular(pi/3),
+         20)
+     y <- c((1 + den) * sin(x), rev(sin(x)))
+     x <- c((1 + den) * cos(x), rev(cos(x)))
+     attr(x, "circularp") <- attr(y, "circularp") <- NULL
+     res <- list(x = x, y = y)
+     return(res)
+ }
```



```
> res <- cont(circular(seq(pi/3 - pi/4, pi/3 +
+     pi/4, 0.01)))
> polygon(x = res, col = 4)
```

Return

```
> A <- function(x, alpha) {
+     if (alpha == -1)
+         a <- log(x + 1)
+     else a <- ((x + 1)^(alpha + 1) - 1)/(alpha +
+         1)
+     return(a)
+ }
> plot(function(x) A(x, alpha = -1/2), from = -1,
+     to = 10, xlab = expression(delta), ylab = expression(
+     col = 4)
> plot(function(x) A(x, alpha = -1), from = -1,
+     to = 10, col = 5, add = TRUE)
> plot(function(x) A(x, alpha = -2), from = -1,
+     to = 10, col = 6, add = TRUE)
> abline(0, 1, col = 3)
```


Focus Session on Robust Statistics is Friday, 3:00pm at HS 0.7

```
> abline(h = 0, lty = 3)
> abline(v = 0, lty = 3)
> legend(6, -0.1, legend = c("ML", "HD", "KL",
+     "NCS"), col = 3:6, lty = rep(1, 4))
```

Return

Focus Session on Robust Statistics is Friday, 3:00pm at HS 0.7

```
> w <- function(x, alpha) {
+     w <- (A(x, alpha) + 1)/(x + 1)
+     w[w > 1] <- 1
+     w[w < 0] <- 0
+     return(w)
+ }
> weight <- function(x, alpha = -1/2) {
+     w(delta(x), alpha)
+ }
> curve.circular(weight, n = 501, join = TRUE,
+     xlim = c(-0.2, 2), ylim = c(0, 2), col = 2,
+     tcl.text = 0.2, cex = 0.9)
> plot.function.circular(function(x) weight(x,
+     alpha = -1), add = TRUE, col = 3)
> plot.function.circular(function(x) weight(x,
```


Focus Session on Robust Statistics is Friday, 3:00pm at HS 0.7

```
+     alpha = -2), add = TRUE, col = 4)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.25, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.5, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.75, 100), lty = 2, col = 1)
> legend(1.3, 1.7, legend = c("HD", "KL", "NCS"),
+     col = 2:4, lty = rep(1, 3))
```

Return

```
> library(wle)
> wvmhd <- wle.vonmises(dir, smooth = 75, group = 5,
+     alpha = -1/2)
> wvmkl <- wle.vonmises(dir, smooth = 75, group = 5,
+     alpha = -1)
> wvmncs <- wle.vonmises(dir, smooth = 75, group = 5,
+     alpha = -2)
> plot(density(dir, bw = 40), ticks = FALSE,
+     xlim = c(-0.8, 1.4), ylim = c(-1, 1.94),
+     main = "", xlab = "", ylab = "", col = 2)
> plot.function.circular(function(x) dvonmises(x,
+     mu = mvm$mu, kappa = mvm$kappa), join = TRUE,
+     add = TRUE, col = 3)
> lines(c(mvm$mu, mvm$mu), c(-1, 0.2), col = 3)
> plot.function.circular(function(x) dvonmises(x,
```



```
+     mu = wvmhd$mu, kappa = wvmhd$kappa), join = TRUE,
+     add = TRUE, col = 4)
> lines(c(wvmhd$mu, wvmhd$mu), c(-1, 0.2), col = 4)
> plot.function.circular(function(x) dvonmises(x,
+     mu = wvmkl$mu, kappa = wvmkl$kappa), join = TRUE,
+     add = TRUE, col = 5)
> lines(c(wvmkl$mu, wvmkl$mu), c(-1, 0.2), col = 5)
> plot.function.circular(function(x) dvonmises(x,
+     mu = wvmncs$mu, kappa = wvmncs$kappa),
+     join = TRUE, add = TRUE, col = 6)
> lines(c(wvmncs$mu, wvmncs$mu), c(-1, 0.2),
+     col = 6)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.5, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(1, 100), lty = 2, col = 1)
```



```
> legend(0.8, 2, legend = c("f*(x)", "MLE", expression(past
+     -1/2, sep = "")), expression(paste("WLE",
+     -1, sep = "")), expression(paste("WLE",
+     -2, sep = ""))), col = 2:6, lty = rep(1,
+     5))
```

Return


```
> library(wle)
> wwnhd <- wle.wrappednormal(dir, smooth = 0.006,
+     group = 5, alpha = -1/2)
> wwnkl <- wle.wrappednormal(dir, smooth = 0.006,
+     group = 5, alpha = -1)
> wwnncs <- wle.wrappednormal(dir, smooth = 0.006,
+     group = 5, alpha = -2)
> plot(density(dir, bw = 40), ticks = FALSE,
+     xlim = c(-0.8, 1.4), ylim = c(-1, 1.94),
+     main = "", xlab = "", ylab = "", col = 2)
> plot.function.circular(function(x) dwrappednormal(x,
+     mu = mwn$mu, rho = mwn$rho), join = TRUE,
+     add = TRUE, col = 3)
> lines(c(mwn$mu, mwn$mu), c(-1, 0.2), col = 3)
> plot.function.circular(function(x) dwrappednormal(x,
```

```
+     mu = wwnhd$mu, rho = wwnhd$rho), join = TRUE,
+     add = TRUE, col = 4)
> lines(c(wwnhd$mu, wwnhd$mu), c(-1, 0.2), col = 4)
> plot.function.circular(function(x) dwrappednormal(x,
+     mu = wwnkl$mu, rho = wwnkl$rho), join = TRUE,
+     add = TRUE, col = 5)
> lines(c(wwnkl$mu, wwnkl$mu), c(-1, 0.2), col = 5)
> plot.function.circular(function(x) dwrappednormal(x,
+     mu = wwnncs$mu, rho = wwnncs$rho), join = TRUE,
+     add = TRUE, col = 6)
> lines(c(wwnncs$mu, wwnncs$mu), c(-1, 0.2),
+     col = 6)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.5, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(1, 100), lty = 2, col = 1)
```



```
> legend(0.8, 2, legend = c("f*(x)", "MLE", expression(past
+     -1/2, sep = "")), expression(paste("WLE",
+     -1, sep = "")), expression(paste("WLE",
+     -2, sep = ""))), col = 2:6, lty = rep(1,
+     5))
```

Return


```
> library(wle)
> dvmhd <- mde.vonmises(dir, bw = 75, n = 1024,
+     alpha = -1/2)
> dvmkl <- mde.vonmises(dir, bw = 75, n = 1024,
+     alpha = -1)
> dvmncs <- mde.vonmises(dir, bw = 75, n = 1024,
+     alpha = -2)
> plot(density(dir, bw = 40), ticks = FALSE,
+     xlim = c(-0.8, 1.4), ylim = c(-1, 1.94),
+     main = "", xlab = "", ylab = "", col = 2)
> plot.function.circular(function(x) dvonmises(x,
+     mu = mvm$mu, kappa = mvm$kappa), join = TRUE,
+     add = TRUE, col = 3)
> lines(c(mvm$mu, mvm$mu), c(-1, 0.2), col = 3)
> plot.function.circular(function(x) dvonmises(x,
```



```
+     mu = dvmhd$mu, kappa = dvmhd$kappa), join = TRUE,
+     add = TRUE, col = 4)
> lines(c(dvmhd$mu, dvmhd$mu), c(-1, 0.2), col = 4)
> plot.function.circular(function(x) dvonmises(x,
+     mu = dvmkl$mu, kappa = dvmkl$kappa), join = TRUE,
+     add = TRUE, col = 5)
> lines(c(dvmkl$mu, dvmkl$mu), c(-1, 0.2), col = 5)
> plot.function.circular(function(x) dvonmises(x,
+     mu = dvmncs$mu, kappa = dvmncs$kappa),
+     join = TRUE, add = TRUE, col = 6)
> lines(c(dvmncs$mu, dvmncs$mu), c(-1, 0.2),
+     col = 6)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.5, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(1, 100), lty = 2, col = 1)
```

```
> legend(0.8, 2, legend = c("f*(x)", "MLE", expression(past
+     -1/2, sep = "")), expression(paste("MDE",
+     -1, sep = "")), expression(paste("MDE",
+     -2, sep = ""))), col = 2:6, lty = rep(1,
+     5))
```

Return


```
> library(wle)
> dwnhd <- mde.wrappednormal(dir, bw = 0.0375,
+     n = 1024, alpha = -1/2)
> dwnkl <- mde.wrappednormal(dir, bw = 0.0375,
+     n = 1024, alpha = -1)
> dwnncs <- mde.wrappednormal(dir, bw = 0.09,
+     n = 1024, alpha = -2)
> plot(density(dir, bw = 40), ticks = FALSE,
+     xlim = c(-0.8, 1.4), ylim = c(-1, 1.94),
+     main = "", xlab = "", ylab = "", col = 2)
> plot.function.circular(function(x) dwrappednormal(x,
+     mu = mwn$mu, rho = mwn$rho), join = TRUE,
+     add = TRUE, col = 3)
> lines(c(mwn$mu, mwn$mu), c(-1, 0.2), col = 3)
> plot.function.circular(function(x) dwrappednormal(x,
```



```
+     mu = dwnhd$mu, rho = dwnhd$rho), join = TRUE,
+     add = TRUE, col = 4)
> lines(c(dwnhd$mu, dwnhd$mu), c(-1, 0.2), col = 4)
> plot.function.circular(function(x) dwrappednormal(x,
+     mu = dwnkl$mu, rho = dwnkl$rho), join = TRUE,
+     add = TRUE, col = 5)
> lines(c(dwnkl$mu, dwnkl$mu), c(-1, 0.2), col = 5)
> plot.function.circular(function(x) dwrappednormal(x,
+     mu = dwnncs$mu, rho = dwnncs$rho), join = TRUE,
+     add = TRUE, col = 6)
> lines(c(dwnncs$mu, dwnncs$mu), c(-1, 0.2),
+     col = 6)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(0.5, 100), lty = 2, col = 1)
> lines(circular(seq(0, 2 * pi, length = 100)),
+     rep(1, 100), lty = 2, col = 1)
```



```
> legend(0.8, 2, legend = c("f*(x)", "MLE", expression(past
+     -1/2, sep = "")), expression(paste("MDE",
+     -1, sep = "")), expression(paste("MDE",
+     -2, sep = ""))), col = 2:6, lty = rep(1,
+     5))
```

Return

```
> bw <- seq(25, 200, 25)
> temp <- function(x, dir, alpha) wle.vonmises(x = dir,
+     group = 5, smooth = x, alpha = alpha)$mu
> muhdbw <- sapply(bw, temp, dir = dir, alpha = -1/2)
> muklbw <- sapply(bw, temp, dir = dir, alpha = -1)
> muncsbw <- sapply(bw, temp, dir = dir, alpha = -2)
> plot(bw, muhdbw, xlab = "bandwidth", ylab = expression(mu
+     main = "", type = "l", col = 4, ylim = range(muhdbw,
+         muklbw, muncsbw))
> lines(bw, muklbw, col = 5)
> lines(bw, muncsbw, col = 6)
> bw <- seq(25, 200, 25)
> temp <- function(x, dir, alpha) wle.vonmises(x = dir,
+     group = 5, smooth = x, alpha = alpha)$kappa
> kappahdbw <- sapply(bw, temp, dir = dir, alpha = -1/2)
```



```
> kappaklbw <- sapply(bw, temp, dir = dir, alpha = -1)
> kappancsbw <- sapply(bw, temp, dir = dir, alpha = -2)
> plot(bw, kappahdbw, xlab = "bandwidth", ylab = expression
+     main = "", type = "l", col = 4, ylim = range(kappahdb
+         kappaklbw, kappancsbw))
> lines(bw, kappaklbw, col = 5)
> lines(bw, kappancsbw, col = 6)
```

Return

C. Agostinelli. *Inferenza statistica robusta basata sulla funzione di verosimiglianza pesata: alcuni sviluppi*. PhD thesis, Dipartimento di Scienze Statistiche, Università di Padova, 1998.

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These slides are prepared using LaTeX, beamer class and Sweave package in R . They are compiled with R ver. 2.2.0 running under OS darwin7.9.0 and packages `circular` ver. 0.3-5, `wle` ver. 0.9-2.