Approximate Conditional-mean Type Filtering for State-space Models

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joint work with

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UseR! 2008, Dortmund, Germany

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Linear State Space Models

- State equation:

$$\boldsymbol{x}_t \;=\; \boldsymbol{\Phi}\boldsymbol{x}_{t-1} + \boldsymbol{\varepsilon}_t$$

- Observation equation:

$$\boldsymbol{y}_t \;=\; \boldsymbol{H}\boldsymbol{x}_t + \boldsymbol{v}_t$$

- Ideal model assumptions:

$$\boldsymbol{x}_0 \sim \mathcal{N}_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)\;, \quad \boldsymbol{\varepsilon}_t \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{Q})\;, \quad \boldsymbol{v}_t \sim \mathcal{N}_q(\mathbf{0}, \boldsymbol{R})\;,$$

all independent

Classical Kalman Filter

- Initialization $(t = 0)$:

$$\boldsymbol{x}_{0|0} = \boldsymbol{\mu}_0 \ , \qquad \boldsymbol{P}_0 = \boldsymbol{\Sigma}_0$$

- Prediction $(t \geq 1)$:

$$\begin{aligned} \boldsymbol{x}_{t|t-1} &= \boldsymbol{\Phi} \boldsymbol{x}_{t-1|t-1} \\ \boldsymbol{M}_t &= \boldsymbol{\Phi} \boldsymbol{P}_{t-1} \boldsymbol{\Phi}^\top + \boldsymbol{Q} = \mathrm{Cov}(\boldsymbol{x}_{t|t-1}) \end{aligned}$$

- Correction $(t \geq 1)$:

$$\begin{aligned} \boldsymbol{x}_{t|t} &= \boldsymbol{x}_{t|t-1} + \boldsymbol{K}_t (\boldsymbol{y}_t - \boldsymbol{H} \boldsymbol{x}_{t|t-1}) \\ \boldsymbol{P}_t &= \boldsymbol{M}_t - \boldsymbol{K}_t \boldsymbol{H} \boldsymbol{M}_t = \mathrm{Cov}(\boldsymbol{x}_{t|t}) \end{aligned}$$

with $\boldsymbol{K}_t = \boldsymbol{M}_t \boldsymbol{H}^\top (\boldsymbol{H} \boldsymbol{M} \boldsymbol{H}^\top + \boldsymbol{R})^{-1}$ (Kalman gain)

Types of Outliers

- Innovational Outliers (IO's):
  - state equation is contaminated
  - not considered here
- Additive Outliers (AO's):
  - observations are contaminated
  - error process v_t is affected
  - possible model:

$$\mathcal{CN}_q(\gamma, \boldsymbol{R}, \boldsymbol{R}_c) = (1-\gamma)\mathcal{N}_q(\mathbf{0}, \boldsymbol{R}) + \gamma\mathcal{N}_q(\boldsymbol{\mu}_c, \boldsymbol{R}_c)$$

- Other Types of Outliers:
  - substitutive outliers (SO's)
  - patchy outliers

Masreliez's Theorem (1975)

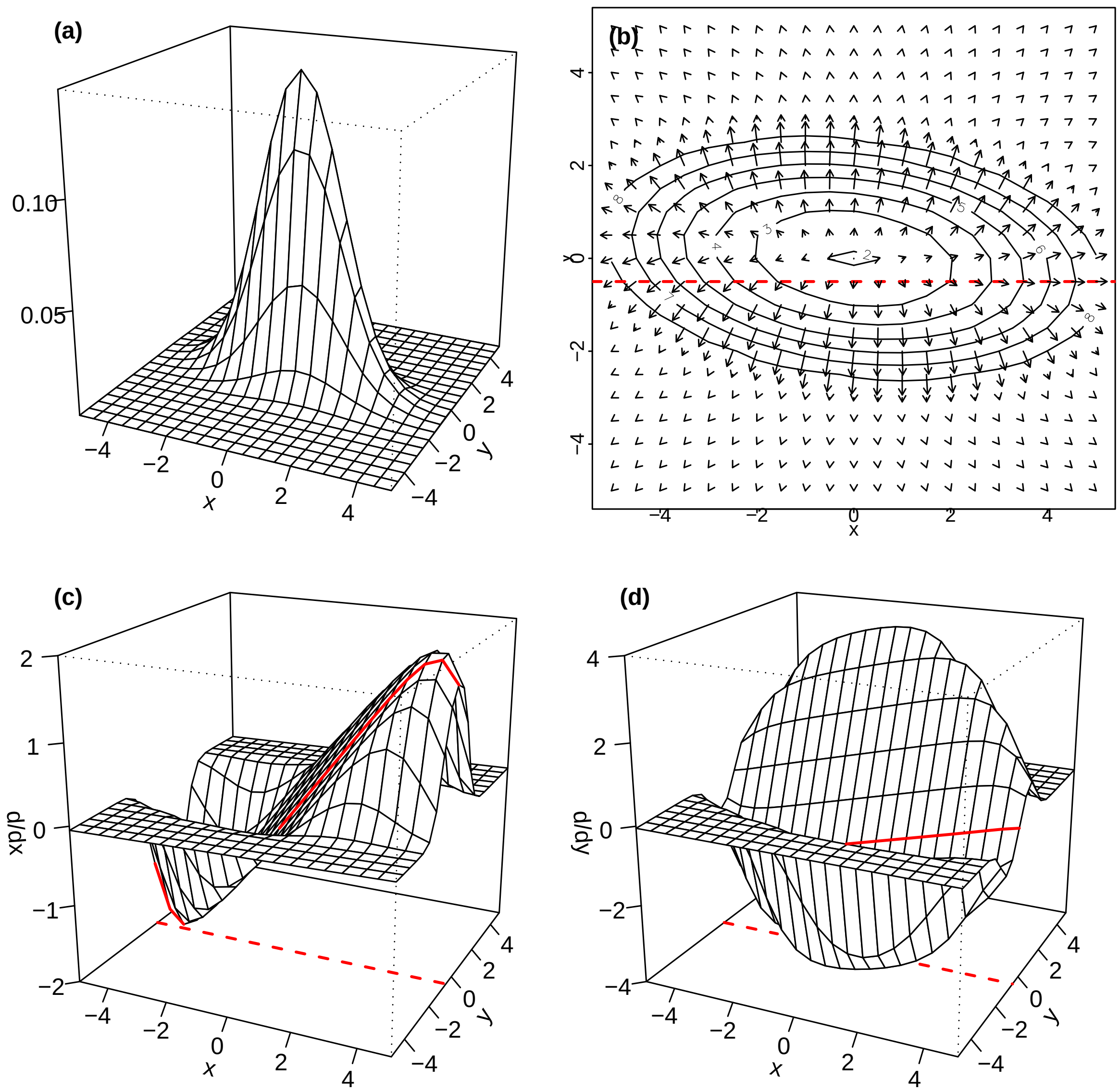
- If $\boldsymbol{x}_t|\boldsymbol{Y}_{t-1} \sim \mathcal{N}_p(\boldsymbol{x}_{t|t-1}, \boldsymbol{M}_t)$, $t \geq 1$, then $\boldsymbol{x}_{t|t} = E(\boldsymbol{x}_t|\boldsymbol{Y}_t)$, $t \geq 1$, is generated by the recursions

$$\begin{aligned} \boldsymbol{x}_{t|t} &= \boldsymbol{x}_{t|t-1} + \boldsymbol{M}_t \boldsymbol{H}^\top \Psi_t(\boldsymbol{y}_t) \\ \boldsymbol{P}_t &= \boldsymbol{M}_t - \boldsymbol{M}_t \boldsymbol{H}^\top \Psi_t'(\boldsymbol{y}_t) \boldsymbol{H} \boldsymbol{M}_t \\ \boldsymbol{M}_{t+1} &= \boldsymbol{\Phi} \boldsymbol{P}_t \boldsymbol{\Phi}^\top + \boldsymbol{Q}\ , \end{aligned}$$

  with $(\Psi_t(\boldsymbol{y}))_i = -(\partial/\partial y_i) \log f_{\boldsymbol{y}_t}(\boldsymbol{y}|\boldsymbol{Y}_{t-1})$ and $(\Psi_t'(\boldsymbol{y}))_{ij} = (\partial/\partial y_j)(\Psi_t(\boldsymbol{y}))_i$.

- $\Psi_t(\boldsymbol{y})$ is called the score function.
- Note: If $f_{\boldsymbol{y}_t}(.|\boldsymbol{Y}_{t-1})$ is Gaussian, Masreliez's filter reduces to the Kalman filter.

The Score Function Ψ_t



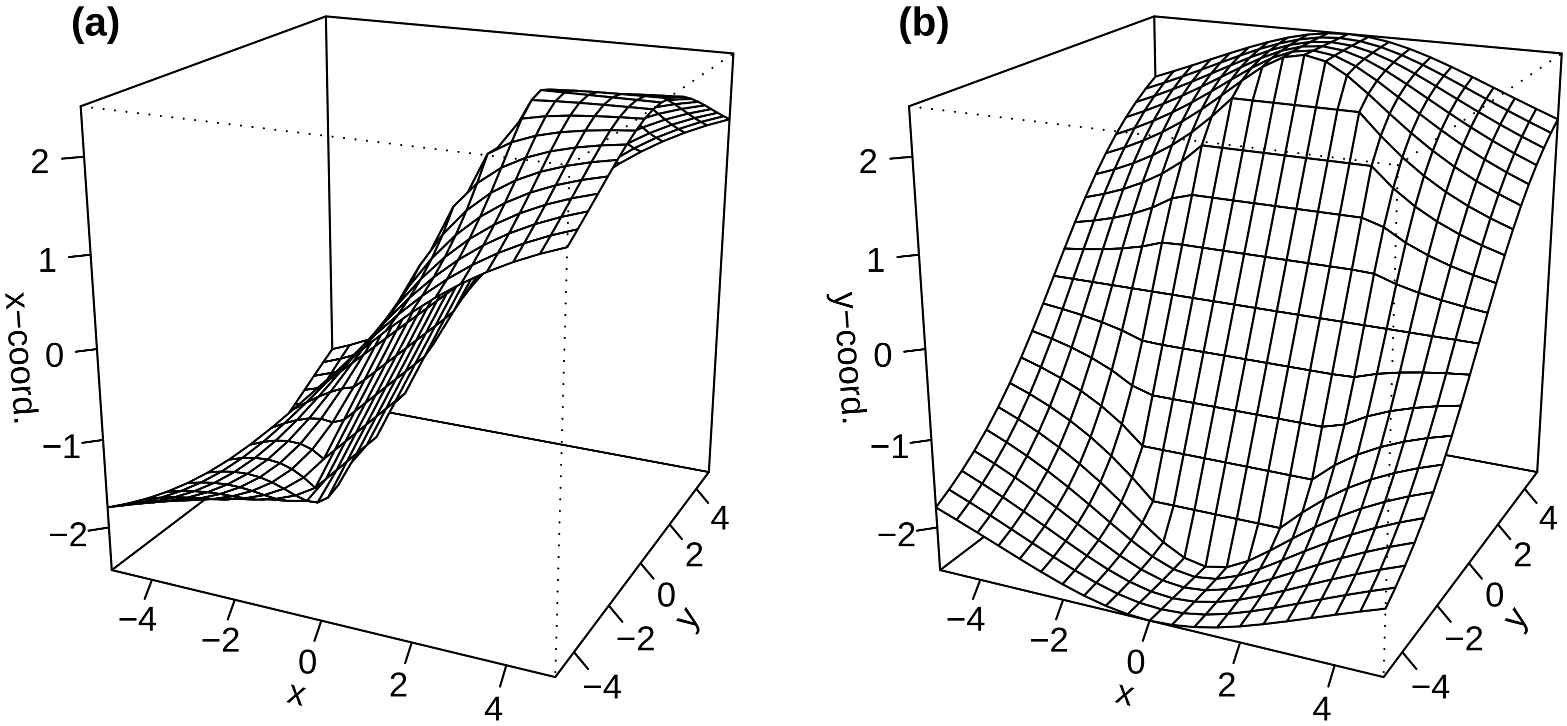
Multivariate ACM-type Filter

- approximate conditional-mean (ACM) type filter
- proposed by B. Spangl and R. Dutter (2008)
- modified correction step:

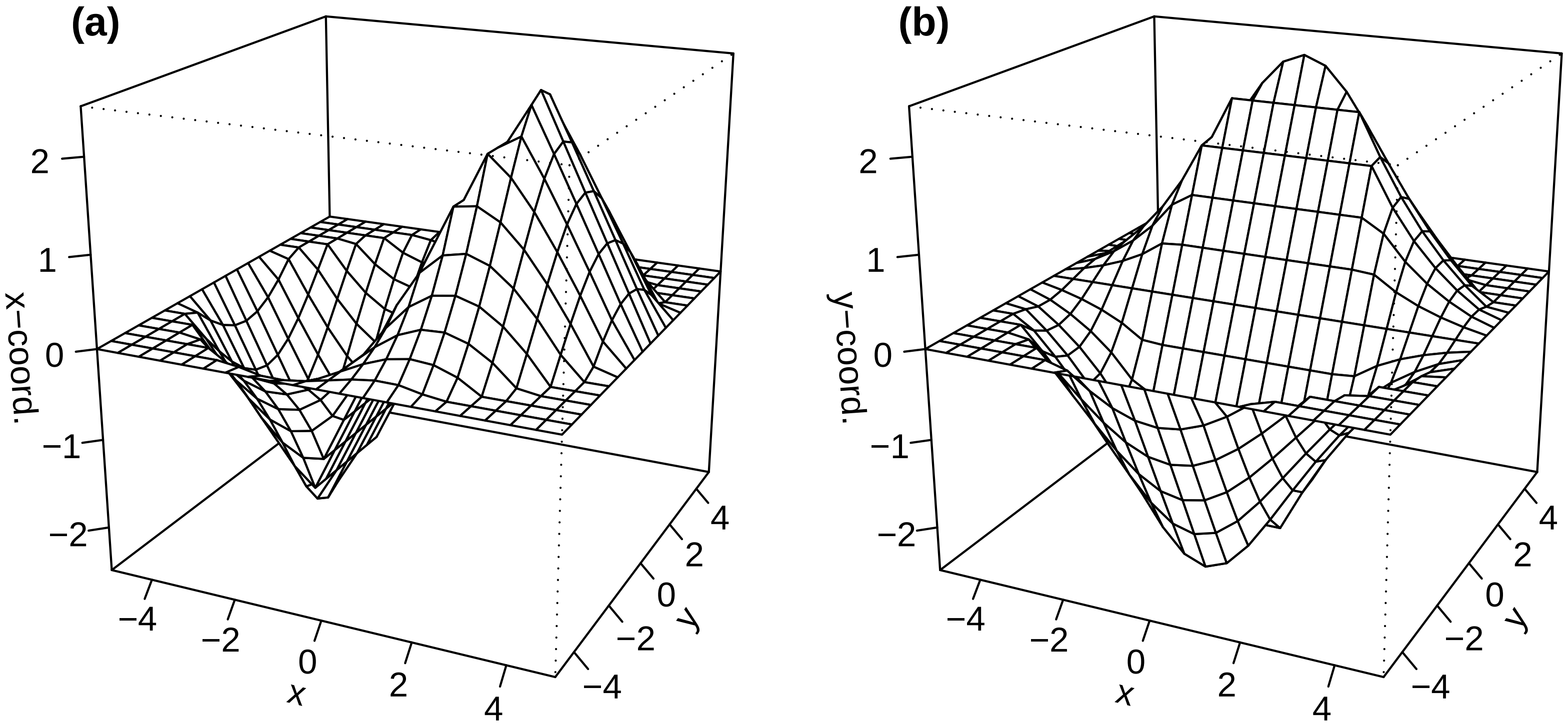
$$\begin{aligned} \boldsymbol{x}_{t|t} &= \boldsymbol{x}_{t|t-1} + \boldsymbol{M}_t \boldsymbol{H}^\top \boldsymbol{S}_t \psi(\boldsymbol{S}_t(\boldsymbol{y}_t - \boldsymbol{H}\boldsymbol{x}_{t|t-1})) \\ \boldsymbol{P}_t &= \boldsymbol{M}_t - \boldsymbol{M}_t \boldsymbol{H}^\top \boldsymbol{S}_t \psi'(\boldsymbol{S}_t(\boldsymbol{y}_t - \boldsymbol{H}\boldsymbol{x}_{t|t-1})) \boldsymbol{S}_t \boldsymbol{H} \boldsymbol{M}_t \end{aligned}$$

  for an $\boldsymbol{S}_t$ and a ψ-function appropriately chosen
- in the case univariate observations
  equivalent to Martin’s ACM type filter (Martin, 1979)

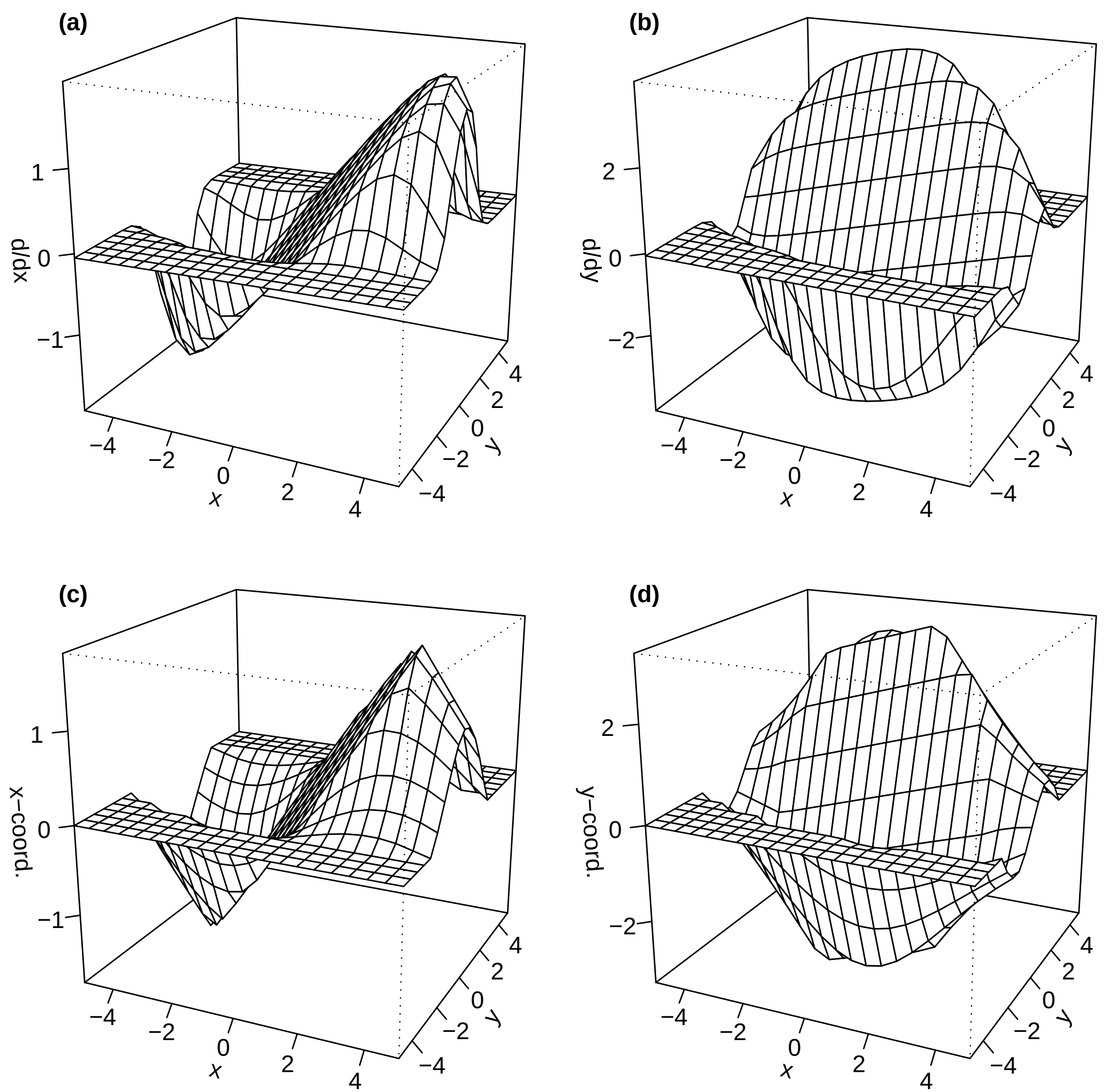
Huber’s Multivariate Psi-function



Hampel's Multivariate Psi-function



Approximating the Score Function



rLS Filter

- proposed by P. Ruckdeschel (2001)
- modified correction step:

$$\boldsymbol{x}_{t|t} \;=\; \boldsymbol{x}_{t|t-1} + \boldsymbol{H}_b(\boldsymbol{K}_t(\boldsymbol{y}_t - \boldsymbol{H}\boldsymbol{x}_{t|t-1}))$$

  with $\boldsymbol{H}_b(\boldsymbol{z}) = \boldsymbol{z}\min\{1, b/\|\boldsymbol{z}\|_2\}$ and $\|.\|_2$ the Euclidean norm
- optimal for SO's in some sense

Simulation

- State Space Process:
  - simulate state space process using two different sets of hyper parameters
  - and AO's from two different contamination setups:

$$\mathcal{N}_2(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}) \quad \text{or} \quad \mathcal{N}_2(\begin{pmatrix} 25 \\ 30 \end{pmatrix}, \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}) .$$

  - vary contamination γ from 0% to 20% by 5%
  - each 400 times
- Filtering:
  - robust filtering (ACM, rLS)
- Evaluation:
  - compare with true state process via MSE

Simulation (cont.)

- Example I:

$$\boldsymbol{\mu_0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{\Phi} = \begin{pmatrix} 0.5 & 0.3 \\ 0.6 & 0.5 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$

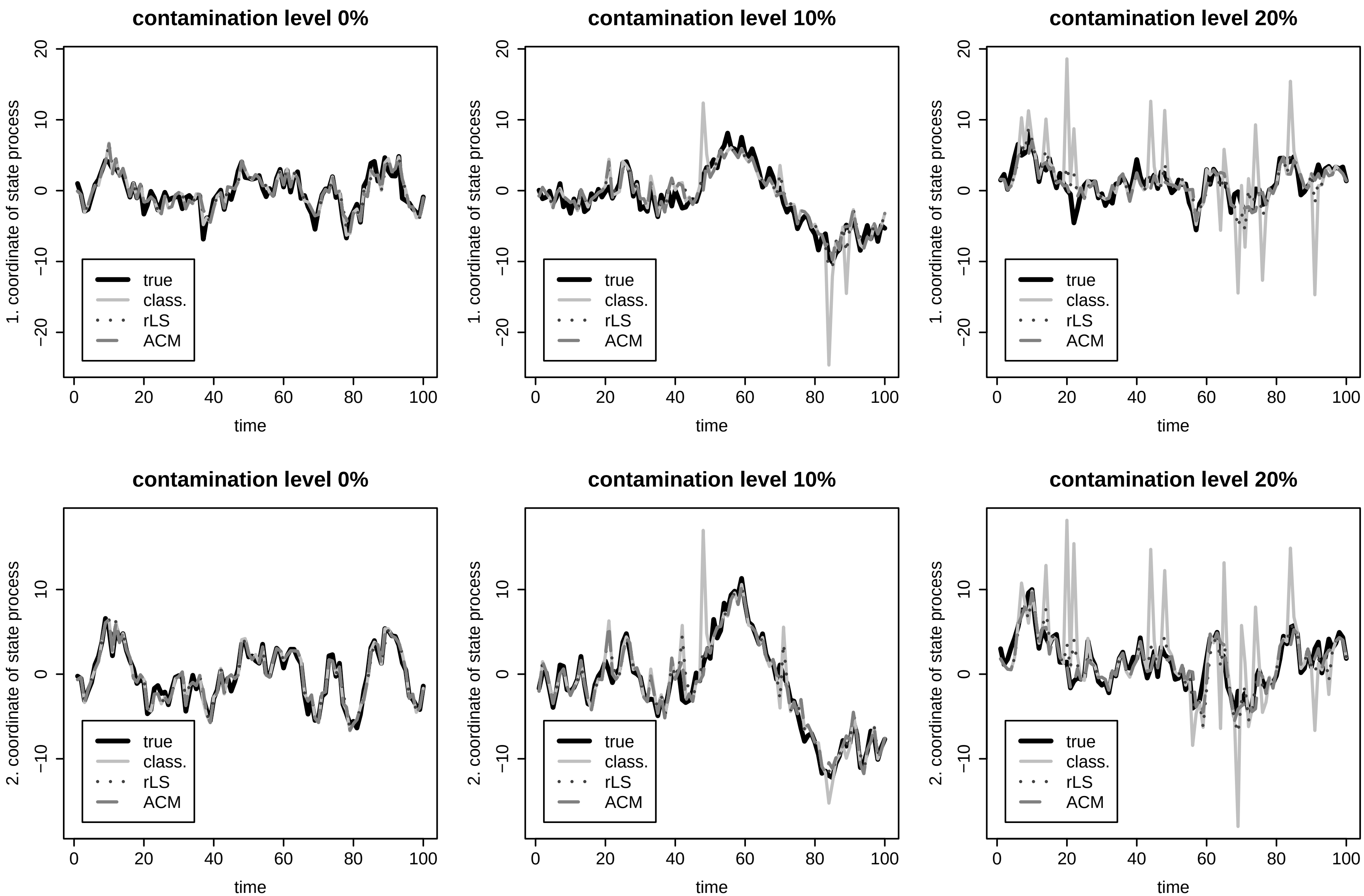
$$\boldsymbol{\Sigma_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{H} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} 2 & -0.2 \\ -0.2 & 0.5 \end{pmatrix}.$$

- Example II:

$$\boldsymbol{\mu_0} = \begin{pmatrix} 20 \\ 0 \end{pmatrix}, \qquad \boldsymbol{\Phi} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix},$$

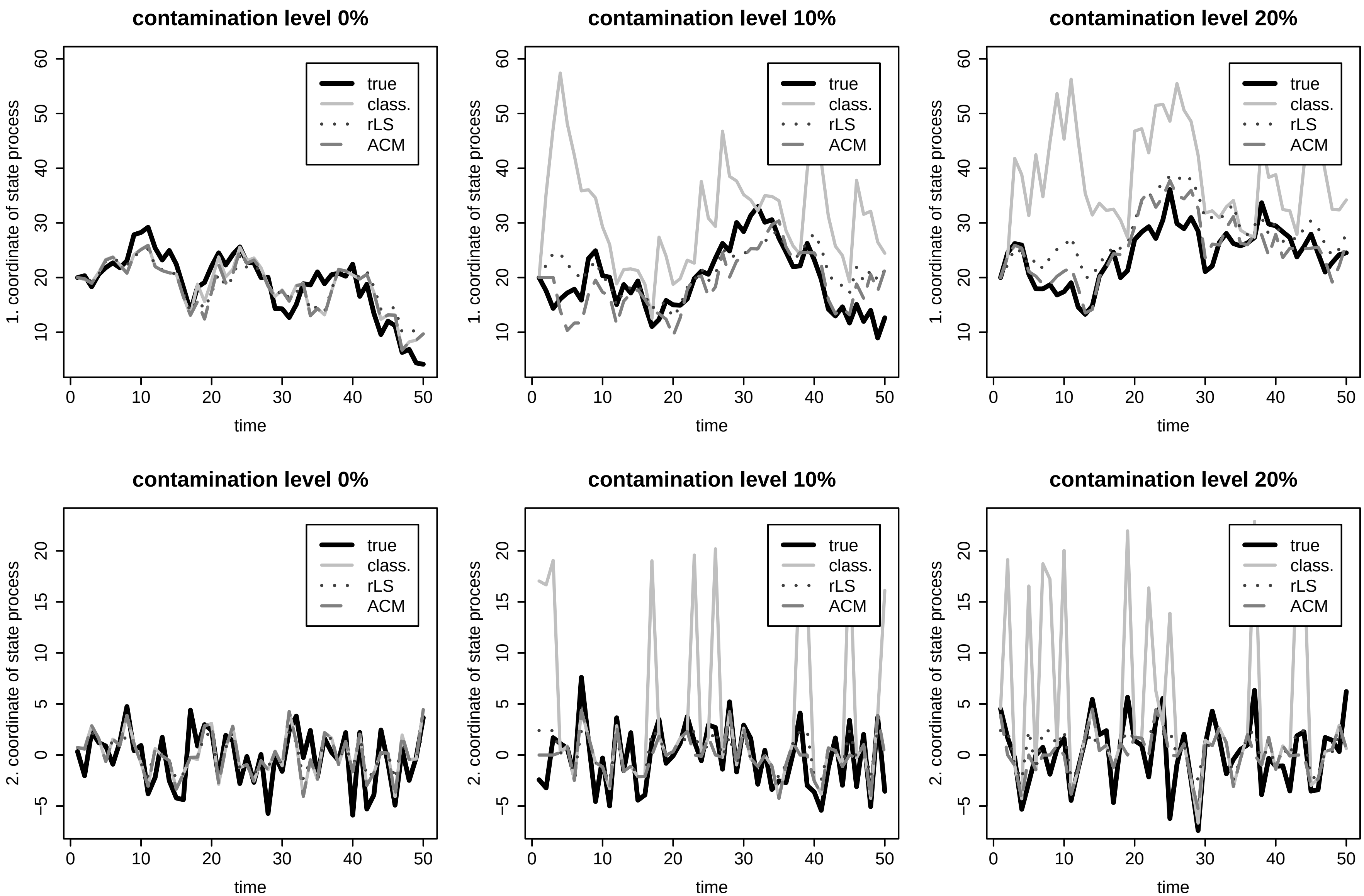
$$\boldsymbol{\Sigma_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{H} = \begin{pmatrix} 0.3 & 1 \\ -0.3 & 1 \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.$$

Results

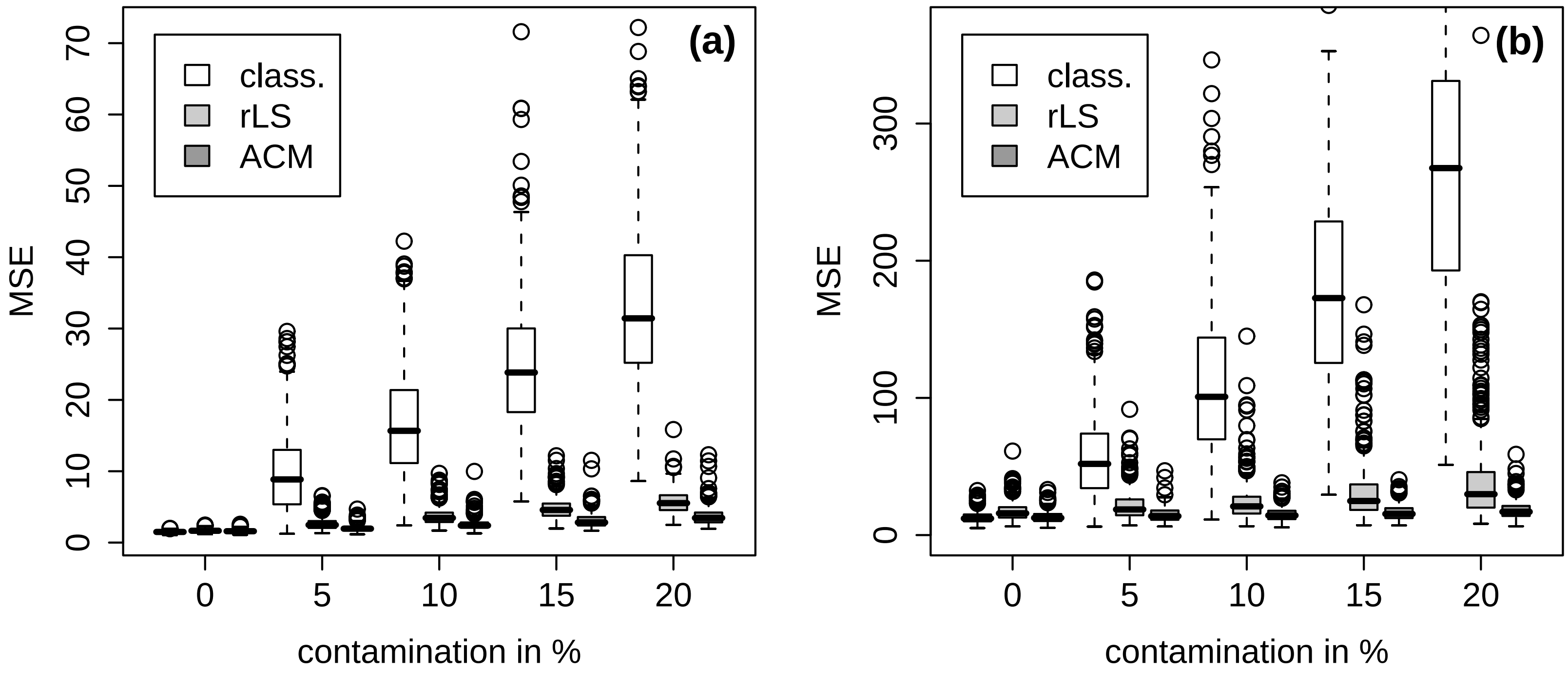


contamination level 0%
contamination level 10%
contamination level 20%
1. coordinate of state process
2. coordinate of state process
time
true
class.
rLS
ACM

Results (cont.)



Results (cont.)



The R package `robKalman`

- general function `recursiveFilter` with parameters:
  - observations
  - state-space model (hyper parameters)
  - functions for the init./pred./corr. step
- available filters:
  - KalmanFilter, rLSFilter, ACMfilter, mACMfilter
  - all: wrappers to `recursiveFilter`

Remarks & Outlook

- ACM performs better than rLS
  for both contamination situations
- rLS yields larger errors in the case of 0% contamination
  because it was calibrated to a loss of efficiency $\delta = 10\%$
- all simulations were made with R
- R-package `robKalman` for filtering already exists
  (*but is still under construction!*)
  `http://r-forge.r-project.org/projects/robkalman/`
- S4 classes for state-space models and filtering results

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