Beta Regression: Shaken, Stirred, Mixed, and Partitioned

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Overview

- Motivation
- Shaken or stirred: Single or double index beta regression for mean and/or precision in **betareg**
- Mixed: Latent class beta regression via **flexmix**
- Partitioned: Beta regression trees via **party**
- Summary

Motivation

Goal: Model dependent variable $y \in (0, 1)$, e.g., rates, proportions, concentrations etc.

Common approach: Model transformed variable $\tilde{y}$ by a linear model, e.g., $\tilde{y} = \text{logit}(y)$ or $\tilde{y} = \text{probit}(y)$ etc.

Disadvantages:

- Model for mean of $\tilde{y}$, not mean of y (Jensen's inequality).
- Data typically heteroskedastic.

Idea: Model y directly using suitable parametric family of distributions plus link function.

Specifically: Maximum likelihood regression model using alternative parametrization of beta distribution (Ferrari & Cribari-Neto 2004).

Beta regression

Beta distribution: Continuous distribution for $0 < y < 1$, typically specified by two shape parameters $p, q > 0$.

Alternatively: Use mean $\mu = p/(p+q)$ and precision $\phi = p + q$.

Probability density function:

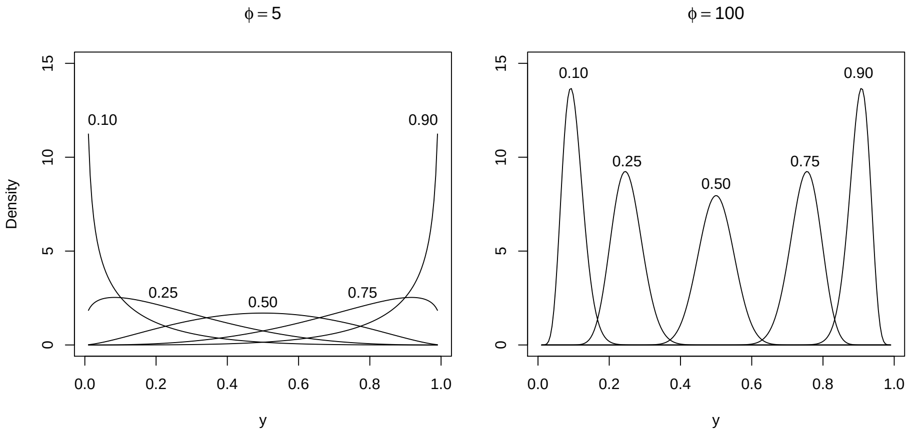
$$
\begin{aligned}
f(y) &= \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\, y^{p-1}\,(1-y)^{q-1} \\
&= \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\,\Gamma((1-\mu)\phi)}\, y^{\mu\phi-1}\,(1-y)^{(1-\mu)\phi-1}
\end{aligned}
$$

where $\Gamma(\cdot)$ is the gamma function.

Properties: Flexible shape. Mean $\mathsf{E}(y) = \mu$ and

$$
\mathsf{Var}(y) = \frac{\mu\,(1-\mu)}{1+\phi}.
$$

Beta regression



Beta regression

Regression model:

- Observations $i = 1, \dots, n$ of dependent variable y_i.
- Link parameters μ_i and ϕ_i to sets of regressor x_i and z_i.
- Use link functions g_1 (logit, probit, ...) and g_2 (log, identity, ...).

$$\begin{aligned} g_1(\mu_i) &= x_i^\top \beta, \\ g_2(\phi_i) &= z_i^\top \gamma. \end{aligned}$$

Inference:

- Coefficients β and γ are estimated by maximum likelihood.
- The usual central limit theorem holds with associated asymptotic tests (likelihood ratio, Wald, score/LM).

Implementation in R

Model fitting:

- Package **betareg** with main model fitting function `betareg()`.
- Interface and fitted models are designed to be similar to `glm()`.
- Model specification via `formula` plus `data`.
- Two part formula, e.g., `y ~ x1 + x2 + x3 | z1 + z2`.
- Log-likelihood is maximized numerically via `optim()`.
- Extractors: `coef()`, `vcov()`, `residuals()`, `logLik()`, ...

Inference:

- Base methods: `summary()`, `AIC()`, `confint()`.
- Methods from **lmtest** and **car**: `lrtest()`, `waldtest()`, `coeftest()`, `linearHypothesis()`.
- Moreover: Multiple testing via **multcomp** and structural change tests via **strucchange**.

Illustration: Reading accuracy

Data: From Smithson & Verkuilen (2006).

- 44 Australian primary school children.
- Dependent variable: Score of test for reading `accuracy`.
- Regressors: Indicator `dyslexia` (yes/no), nonverbal `iq` score.

Analysis:

- OLS for transformed data leads to non-significant effects.
- OLS residuals are heteroskedastic.
- Beta regression captures heteroskedasticity and shows significant effects.

Illustration: Reading accuracy

```
> data("ReadingSkills", package = "betareg")
> rs_ols <- lm(qlogis(accuracy) ~ dyslexia * iq,
+   data = ReadingSkills)
> coeftest(rs_ols)

t test of coefficients:

            Estimate Std. Error t value  Pr(>|t|)
(Intercept)  1.60107    0.22586  7.0888 1.411e-08 ***
dyslexia    -1.20563    0.22586 -5.3380 4.011e-06 ***
iq           0.35945    0.22548  1.5941   0.11878
dyslexia:iq -0.42286    0.22548 -1.8754   0.06805 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> bptest(rs_ols)

studentized Breusch-Pagan test

data:  rs_ols
BP = 21.692, df = 3, p-value = 7.56e-05
```

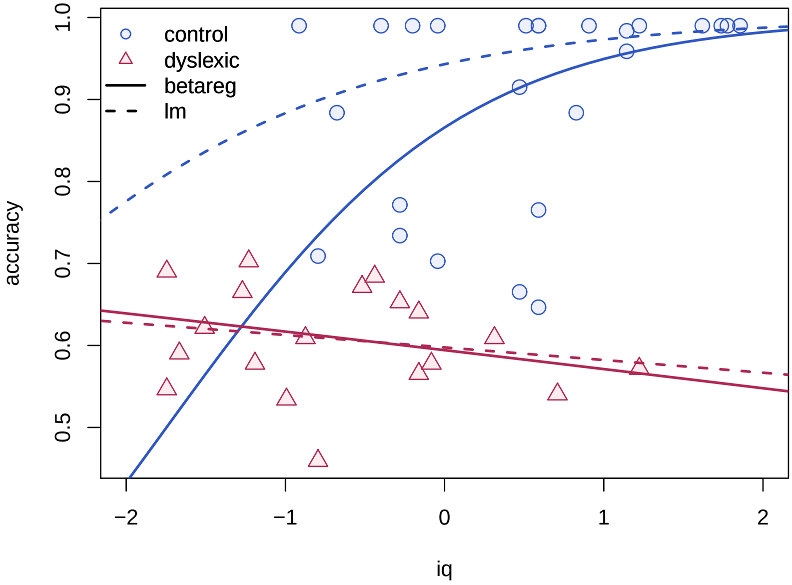
Illustration: Reading accuracy

```
> rs_beta <- betareg(accuracy ~ dyslexia * iq | dyslexia + iq,
+   data = ReadingSkills)
> coeftest(rs_beta)

z test of coefficients:

                  Estimate Std. Error z value  Pr(>|z|)
(Intercept)        1.12323    0.14283  7.8638 3.725e-15 ***
dyslexia          -0.74165    0.14275 -5.1952 2.045e-07 ***
iq                 0.48637    0.13315  3.6528 0.0002594 ***
dyslexia:iq       -0.58126    0.13269 -4.3805 1.184e-05 ***
(phi)_(Intercept)  3.30443    0.22274 14.8353 < 2.2e-16 ***
(phi)_dyslexia     1.74656    0.26232  6.6582 2.772e-11 ***
(phi)_iq           1.22907    0.26720  4.5998 4.228e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Illustration: Reading accuracy



Extensions: Partitions and mixtures

So far: Reuse standard inference methods for fitted model objects.

Now: Reuse fitting functions in more complex models.

Model-based recursive partitioning: Package **party**.

- Idea: Recursively split sample with respect to available variables.
- Aim: Maximize partitioned likelihood.
- Fit: One model per node of the resulting tree.

Latent class regression, mixture models: Package **flexmix**.

- Idea: Capture unobserved heterogeneity by finite mixtures of regressions.
- Aim: Maximize weighted likelihood with k components.
- Fit: Weighted combination of k models.

Beta regression trees

Partitioning variables: `dyslexia` and further random noise variables.

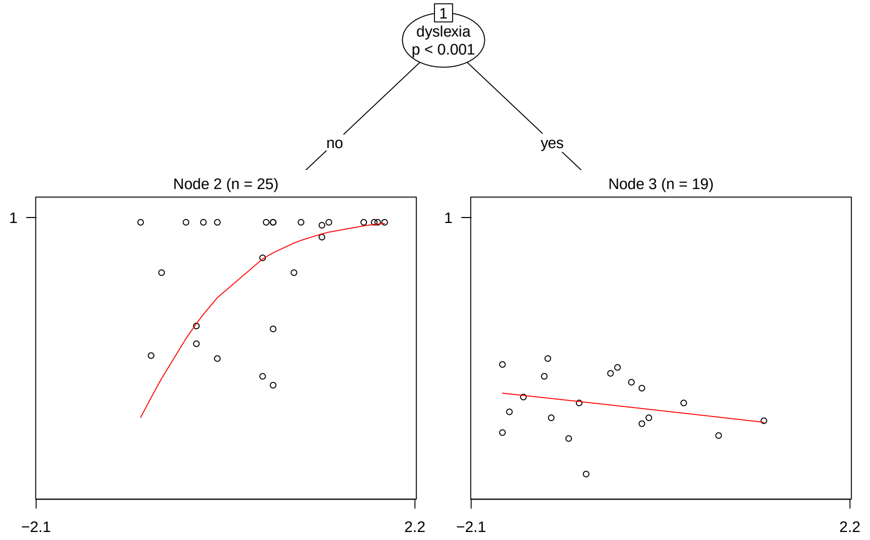
```
> set.seed(1071)
> ReadingSkills$x1 <- rnorm(nrow(ReadingSkills))
> ReadingSkills$x2 <- runif(nrow(ReadingSkills))
> ReadingSkills$x3 <- factor(rnorm(nrow(ReadingSkills)) > 0)
```

Fit beta regression tree: In each node `accuracy`'s mean and precision depends on `iq`, partitioning is done by `dyslexia` and the noise variables `x1`, `x2`, `x3`.

```
> rs_tree <- betatree(accuracy ~ iq | iq,
+   ~ dyslexia + x1 + x2 + x3,
+   data = ReadingSkills, minsplit = 10)
> plot(rs_tree)
```

Result: Only relevant regressor `dyslexia` is chosen for splitting.

Beta regression trees



Latent class beta regression

Setup:

- No `dyslexia` information available.
- Look for $k = 3$ clusters: Two different relationships of type `accuracy ~ iq`, plus component for ideal score of 0.99.

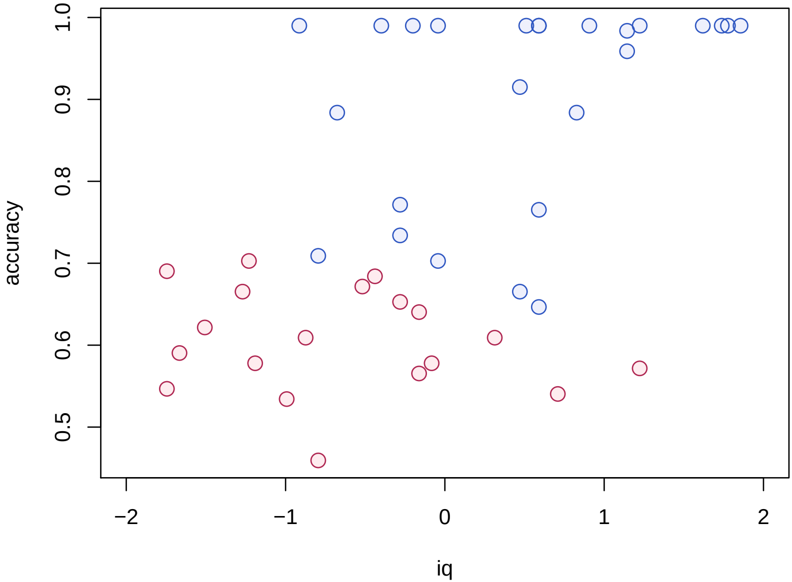
Fit beta mixture regression:

```
> rs_mix <- betamix(accuracy ~ iq, data = ReadingSkills, k = 3,
+   nstart = 10, extra_components = extraComponent(
+   type = "uniform", coef = 0.99, delta = 0.01))
```

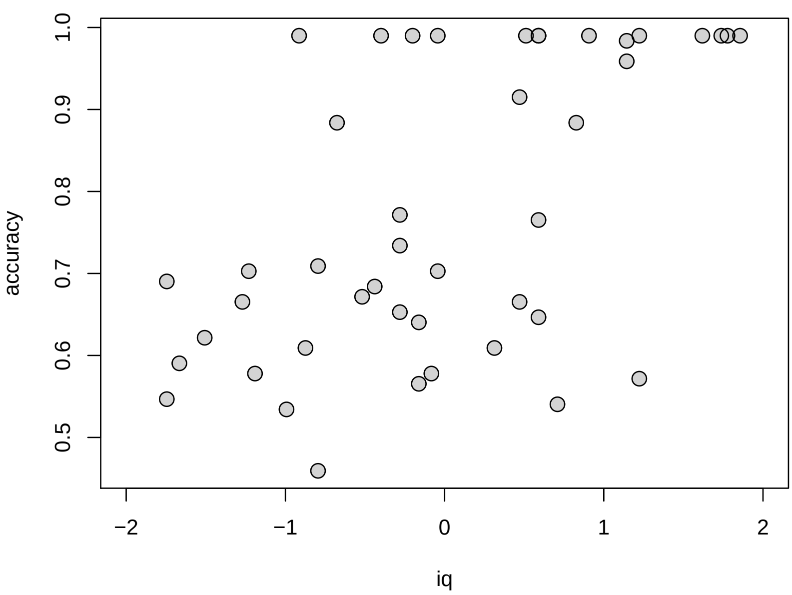
Result:

- Dyslexic children separated fairly well.
- Other children are captured by mixture of two components: ideal reading scores, and strong dependence on `iq` score.

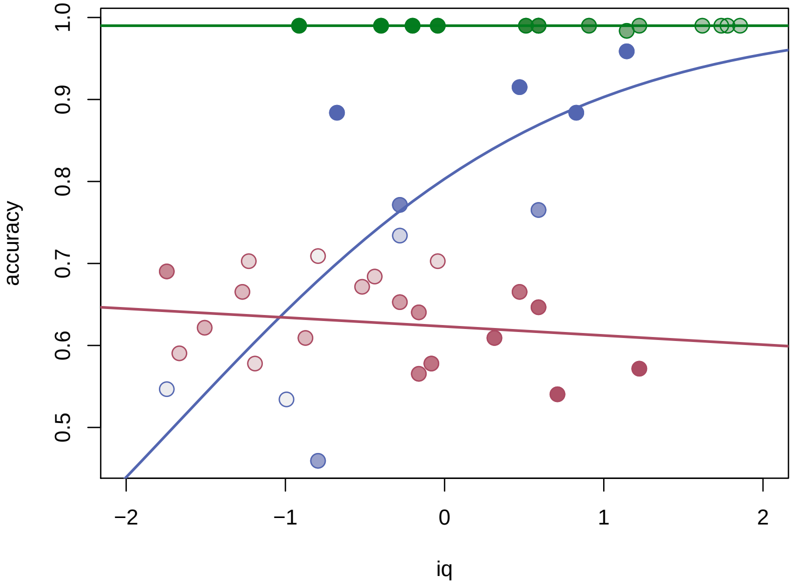
Latent class beta regression



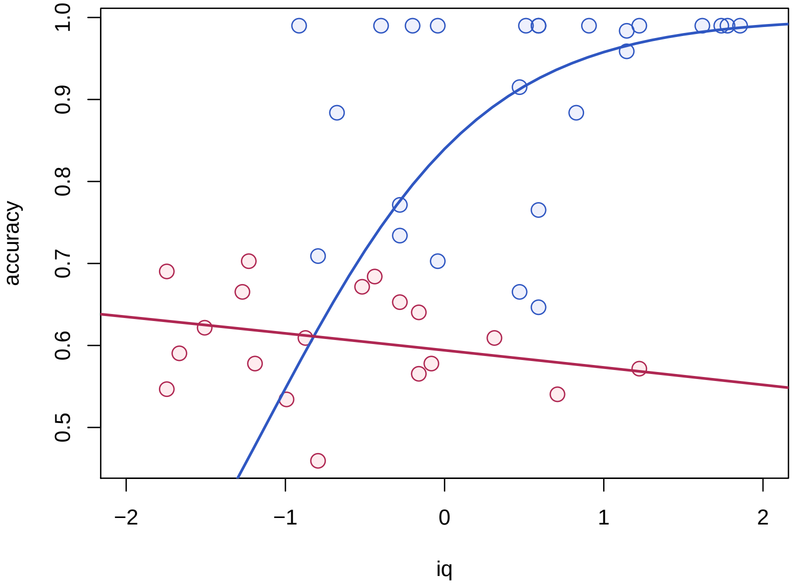
Latent class beta regression



Latent class beta regression



Latent class beta regression



Computational infrastructure

Model-based recursive partitioning:

- **party** provides the recursive partitioning.
- **betareg** provides the models in each node.
  - Model-fitting function: `betareg.fit()` (conveniently without formula processing).
  - Extractor for empirical estimating functions (aka scores or case-wise gradient contributions): `estfun()` method.
  - Some additional (and somewhat technical) S4 glue…

Latent class regression, mixture models:

- **flexmix** provides the E-step for the EM algorithm.
- **betareg** provides the M-step.
  - Model-fitting function: `betareg.fit()`.
  - Extractor for case-wise log-likelihood contributions: `dbeta()`.
  - Some additional (and somewhat more technical) S4 glue…

Summary

Beta regression and extensions:

- Flexible regression model for proportions, rates, concentrations.
- Can capture skewness and heteroskedasticity.
- R implementation **betareg**, similar to `glm()`.
- Due to design, standard inference methods can be reused easily.
- Fitting functions can be plugged into more complex fitters.
- Convenience interfaces available for: Model-based partitioning, finite mixture models.

References

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Bettina Grün and Friedrich Leisch (2008). "FlexMix Version 2: Finite Mixtures with Concomitant Variables and Varying and Constant Parameters." *Journal of Statistical Software*, **28**(4), 1–35. `http://www.jstatsoft.org/v28/i04/`

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